Mathematical Gauge Theory 2 Problem Sheets

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Contents

Problem Sheet 1

Clifford Modules

Exercise 1.1 (The Standard Clifford Module of $\mathbb{R}^4$)**.** Let e_0, e_1, e_2, e_3 be an orthonormal basis of $\mathbb{R}^4$ with the standard Euclidean scalar product. Define a linear map $\gamma : \mathbb{R}^4 \to \operatorname{End}\left(\mathbb{C}^4\right)$ by

$$\gamma\left(e_j\right) = A_j = \begin{pmatrix} 0 & -B_j^\dagger \\ B_j & 0 \end{pmatrix}$$

where

$$B_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Prove that $\left(\mathbb{C}^4, \gamma\right)$ with the standard Hermitian scalar product is a Clifford module for $(\mathbb{R}^4, g_{\text{can}})$.

Exercise 1.2 (Clifford Multiplication with Forms)**.** Let $\left(\mathbb{C}^4, \gamma\right)$ be the standard Clifford module for $(\mathbb{R}^4, g_{\text{can}})$ from Exercise 1.1. Prove that:

(1)

$$\gamma\left(e_0 \wedge e_1 \wedge e_2 \wedge e_3\right) = \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

(2) Under γ the self-dual two-forms

$$e_0 \wedge e_1 + e_2 \wedge e_3$$
$$e_0 \wedge e_2 - e_1 \wedge e_3$$
$$e_0 \wedge e_3 + e_1 \wedge e_2$$

act non-trivially on $\mathbb{C}^2_+$ as $2B_1, 2B_2$ and $2B_3$, respectively, and are zero on $\mathbb{C}^2_-$.

(3) The map γ induces isomorphisms

$$\left(\Lambda^1\left(\mathbb{R}^4\right) \oplus \Lambda^3\left(\mathbb{R}^4\right)\right) \otimes \mathbb{C} \cong \operatorname{Hom}\left(\mathbb{C}^2_+, \mathbb{C}^2_-\right) \oplus \operatorname{Hom}\left(\mathbb{C}^2_-, \mathbb{C}^2_+\right)$$
$$\Lambda^2_\pm\left(\mathbb{R}^4\right) \otimes \mathbb{C} \cong \operatorname{End}_0\left(\mathbb{C}^2_\pm\right)$$
$$\Lambda^4\left(\mathbb{R}^4\right) \otimes \mathbb{C} \cong \mathbb{C} \cdot \operatorname{Id}_{\mathbb{C}^2_\pm},$$

where End_0 denotes the trace-free endomorphisms.

Exercise 1.3 (Schur's Lemma)**.** Let (V, γ) be an irreducible Clifford module. Prove that every automorphism of (V, γ), i.e. every isomorphism $f : V \to V$ of Clifford modules, is of the form $f(\phi) = \lambda\phi$ for some constant $\lambda \in S^1$.

Exercise 1.4 (Anti-linear automorphisms I)**.** Let $J : V \to V$ be a complex anti-linear automorphism of a standard (i.e. irreducible) Clifford module. Show that $J^2 = \pm \operatorname{Id}_V$.

Problem Sheet 2

Spin Groups

Exercise 2.1 (Anti-linear automorphism II)**.** Let $J, J' : V \to V'$ be complex anti-linear isomorphisms between standard Clifford modules. Show that there exists a number $\lambda \in S^1$ so that $J'(\lambda\phi) = \lambda J(\phi)$ for all $\phi \in V$.

Exercise 2.2 (The group $\mathrm{Spin}(n)$)**.** Let $J_0 : \mathbb{C}^N \to \mathbb{C}^N$ be a complex anti-linear automorphism of the standard Clifford module γ_0. The group $\mathrm{Spin}(n)$ is defined as the set of pairs $(\tau, \sigma) \in \mathrm{Spin}^c(n)$ so that σ commutes with J_0. Prove that the homomorphism

$$\begin{aligned} q : \mathrm{Spin}(n) &\to \mathrm{SO}(n) \\ (\tau, \sigma) &\mapsto \tau \end{aligned}$$

is surjective with kernel $\{(I_n, \pm I_N)\} \cong \mathbb{Z}_2$.

Exercise 2.3 ($\mathrm{Spin}^c(n)$ reconstructed from $\mathrm{Spin}(n)$)**.** We consider the quotient

$$\left(\mathrm{Spin}(n) \times S^1\right)/\mathbb{Z}_2,$$

where (τ, σ, λ) gets identified with $(\tau, -\sigma, -\lambda)$. Prove that the homomorphism

$$\begin{aligned} \left(\mathrm{Spin}(n) \times S^1\right)/\mathbb{Z}_2 &\longrightarrow \mathrm{Spin}^c(n) \\ [\tau, \sigma, \lambda] &\longmapsto (\tau, \lambda\sigma) \end{aligned}$$

is an isomorphism.

Exercise 2.4 (Fundamental representation of $\mathrm{SU}(2)$)**.** Show that there exists a fixed matrix $M \in \mathrm{SU}(2)$ such that

$$MAM^\dagger = \overline{A} \quad \forall A \in \mathrm{SU}(2).$$

Problem Sheet 3

$\mathrm{Spin}^c(n)$-bundles

Exercise 3.1 (Associated $\mathrm{Spin}^c(n)$-bundles, cf. Lemma 2.36)**.** Let $Q \to X$ be a principal $\mathrm{Spin}^c(n)$ bundle with an isomorphism $Q/S^1 \cong \mathrm{Fr}(H)$ for an oriented Euclidean vector bundle $H \to X$. Consider the vector bundle $V \to X$ associated to Q by the standard representation

$$\begin{aligned} \mathrm{Spin}^c(n) &\to \mathrm{U}(N) \\ (\tau, \sigma) &\mapsto \sigma \end{aligned}$$

Show that the standard Clifford module γ_0 induces a Clifford module γ for V.

Exercise 3.2 (Twisting of Spin^c-structures I, cf. Lemma 2.20)**.** Let $(V_{\mathfrak{s}}, \gamma_{\mathfrak{s}})$ be a Spin^c-structure and L a line bundle. Show that the pair $(V_{\mathfrak{s}'}, \gamma_{\mathfrak{s}'})$ has a Clifford module structure, where $V_{\mathfrak{s}'} := V_{\mathfrak{s}} \otimes L_\delta$, $i : \mathrm{End}\,(V_{\mathfrak{s}}) \to \mathrm{End}\,(V_{\mathfrak{s}'})$ is an isomorphism, and $\gamma_{\mathfrak{s}'} = i \circ \gamma_{\mathfrak{s}}$.

Exercise 3.3 (Twisting of Spin^c-structures II)**.** Let $\mathfrak{s}$ be a Spin^c-structure on an oriented Riemannian 4-manifold X and E a complex line bundle over X. Show that for the Spin^c-structure $\mathfrak{s}' = \mathfrak{s} \otimes L$ the characteristic line bundle satisfies

$$L_{\mathfrak{s}'} = L_{\mathfrak{s}} \otimes L^2$$

where $L^2 = L \otimes L$.

Exercise 3.4 (Hodge Dual and Differential of 1-forms)**.** Let (X^n, g) be an oriented Riemannian manifold with Levi-Civita connection ∇ and $\eta \in \Omega^1(X)$ a 1-form. Let $p \in X$ and $e_1, \ldots, e_n$ an oriented local frame for TX on an open neighbourhood around p such that

$$\left(\nabla e_i\right)(p) = 0 \quad \forall i \in \{1, \ldots, n\}.$$

Prove that in p

$$*d * \eta = \sum_{i=1}^{n} L_{e_i} \eta\left(e_i\right),$$

where $*$ denotes the Hodge star.

Problem Sheet 4

Topology of Product 4-Manifolds

Exercise 4.1 (Product of Surfaces)**.** The Euler characteristic of a closed n-manifold M is defined by

$$\chi(M) = \sum_{i=0}^{n} (-1)^i b_i(M)$$

where $b_i(M)$ are the Betti numbers. The Euler characteristic is multiplicative for products of manifolds, that is, $\chi(M \times N) = \chi(M) \cdot \chi(N)$. Let $M = \Sigma_g \times \Sigma_h$, where Σ_g, Σ_h are surfaces of genus g and h.

(1) Calculate $\chi(M)$.

(2) Determine bases for all integral homology groups $H_*(M; \mathbb{Z})$ of M and calculate the Betti numbers. Compare with (1).

[Hint: You can use the standard basis of $H_1(\Sigma_g; \mathbb{Z})$ represented by $2g$ embedded circles.]

(3) Determine the intersection form of M.

Exercise 4.2 (Resolving Transverse Intersections)**.** Let M be a smooth 4-manifold and Σ_1, Σ_2 embedded surfaces in M of genus g_1, g_2. Suppose that Σ_1, Σ_2 have precisely one transverse intersection.

(1) Show that by resolving the transverse intersection between Σ_1 and Σ_2 (cf. the proof of Lemma 3.5) we get an embedded surface Σ in M of genus $g_1 + g_2$.

(2) Prove that in $H_2(M; \mathbb{Z})$

$$[\Sigma] = [\Sigma_1] + [\Sigma_2]$$

by arguing that the difference of the classes on both sides is the boundary of a (singular) 3-cycle.

Exercise 4.3 (Intersection Form of Connected Sums)**.** Let M_1 and M_2 be closed, oriented, connected 4-manifolds with intersection forms Q_{M_1} and Q_{M_2} respectively. Show that the intersection form of their connected sum $M_1 \# M_2$ is given by

$$Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2}.$$

Exercise 4.4 (Embedded Surfaces in $S^2 \times S^2$)**.** Let $M = S^2 \times S^2$ and consider the homology classes $a, b \in H_2(M; \mathbb{Z})$ defined by

$$a = \left[S^2 \times \{p\}\right], \quad b = \left[\{q\} \times S^2\right],$$

where $p, q \in S^2$ are arbitrary points.

(1) Prove that the class na for every $n \in \mathbb{Z}$ can be represented by an embedded sphere.

(2) Prove that the class $na + mb$ for every $n, m \in \mathbb{Z} \backslash \{0\}$ can be represented by an embedded surface Σ of genus

$$g = (|n| - 1)(|m| - 1).$$

Problem Sheet 5

Invariants and Embeddings in 4-Manifolds

Exercise 5.1 (Invariants of 4-manifolds)**.** Let M and N be two closed, connected, simply connected, oriented, smooth 4-manifolds.

(1) Prove that M and N are homeomorphic if and only if the the following invariants agree:

- Euler characteristic χ
- signature σ
- parity (even or odd) of the intersection form.

(2) Determine a simple 4-manifold homeomorphic to $M\#\overline{\mathbb{CP}}^2$.

(3) Assume $\sigma(M) = -\sigma(N)$ and even intersection forms Q_M, Q_N. Determine a 4-manifold homeomorphic to $M\#N$.

Exercise 5.2 (Embedded surfaces in $\mathbb{CP}^2$)**.** A projective line is a linear $\mathbb{CP}^1$ in $\mathbb{CP}^2$ (coming from a linear subspace $\mathbb{C}^2 \subset \mathbb{C}^3$). Let $d \geqslant 0$ be a natural number.

(1) We call d projective lines in $\mathbb{CP}^2$ in general position if all intersections between them are transverse and if at most two projective lines intersect in a given point p for all $p \in \mathbb{CP}^2$. Prove that there exists d projective lines in $\mathbb{CP}^2$ in general position for all $d \geqslant 0$.

(2) Determine a smooth surface representing the class $d\left[\mathbb{CP}^1\right] \in H_2(\mathbb{CP}^2; \mathbb{Z})$. What is its genus?

Exercise 5.3 (The Double of a Manifold)**.** Let M be a compact oriented manifold with non-empty boundary and let $\overline{M}$ denote the same manifold with the opposite orientation. The double of M is obtained by gluing together M and $\overline{M}$ along the boundary via the identity map:

$$M \bigcup_{\partial M = \partial \overline{M}} \overline{M}.$$

Now let $D(e)$ denote the disc bundle with Euler class e over S^2. Show that the double of $D(e)$ is diffeomorphic to $S^2 \times S^2$ if and and only if e is even and to $S^2 \widetilde{\times} S^2$ if and only if e is odd.

Exercise 5.4 (Complete Intersections)**.** Let $d = (d_1, d_2, \ldots, d_r)$ be an r-tuple of natural numbers and consider the intersection of r smooth hypersurfaces X_{d_i} of degree d_i in $\mathbb{CP}^{r+2}$:

$$S_d = X_{d_1} \cap X_{d_2} \ldots \cap X_{d_r}.$$

We assume that for all $k = 2, \ldots, r$ the hypersurface X_{d_k} intersects $X_{d_1} \cap \ldots \cap X_{d_{k-1}}$ transversely. Then S_d is a smooth complex surface, called a complete intersection of multidegree d.

(1) Suppose submanifolds M and N of a manifold W intersect transversely. Show that the normal bundles in W are related by $\nu(M \cap N) = \nu(M)|_{M \cap N} \oplus \nu(N)|_{M \cap N}$.

(2) Calculate the Chern classes $c_1\left(S_d\right)$ and $c_2\left(S_d\right)$.

(3) Determine those multidegrees d for which S_d is a $K3$ surface.

[Cheat: you may use without proof that all complete intersections are simply connected.]

Problem Sheet 6

Fiber Bundles and Spinor Identities

Exercise 6.1 (S^2-bundles over Σ_g)**.** Let Σ_g denote the surface of genus g.

(1) Suppose $D^2 \subset \Sigma_g$ is a small disk around a point. Show that $\Sigma_g \backslash D^2$ is homotopy equivalent to a 1-point union $\bigvee_{i=1}^{2g} S_i^1$ of $2g$ circles.

(2) Prove that for every $g \geqslant 0$ there are at most two orientable S^2-bundles over Σ_g up to diffeomorphism.

Exercise 6.2.

(1) Prove that if $P : H_1 \to H_2$ is a Fredholm operator between Hilbert spaces, then coker $P \cong \ker P^*$.

(2) Let M be a closed oriented connected Riemannian manifold and let d^* be the formal adjoint of d with respect to the L^2 scalar product. Show that the operator

$$P = \mathrm{d} + \mathrm{d}^* : \bigoplus_{k \text{ even}} \Omega^k(M) \to \bigoplus_{k \text{ odd}} \Omega^k(M)$$

is Fredholm by showing that $\dim \ker(P)$ and $\dim \operatorname{coker}(P)$ are finite. Compute $\operatorname{ind}(P)$.

Exercise 6.3 (Spinor Identities)**.** Let $\Gamma(V_+)$ be the space of positive spinors associated to a $\operatorname{Spin}^c(n)$ structure on a smooth closed oriented Riemannian 4-manifold. We give $\operatorname{End}(V_+)$ the scalar product

$$\langle A, B \rangle = \operatorname{tr}\left(AB^\dagger\right)$$

and define the quadratic form σ as in the notes:

$$\begin{aligned} \sigma : \Gamma(V_+) &\longrightarrow \Omega_+^2(X, i\mathbb{R}) \\ \Phi &\longmapsto \sigma(\Phi, \Phi) = \gamma^{-1}\left(\left(\Phi \otimes \Phi^\dagger\right)_0\right). \end{aligned}$$

Prove for all $\omega, \eta \in i\Lambda_+^2$ and $\Phi \in V_+$ the following identities:

$$\begin{aligned} \langle \gamma(\omega), \gamma(\eta) \rangle &= 4\langle \omega, \eta \rangle \\ \langle \gamma(\omega)\Phi, \Phi \rangle &= 4\langle \omega, \sigma(\Phi, \Phi) \rangle \\ |\Phi|^4 &= 8|\sigma(\Phi, \Phi)|^2. \end{aligned}$$

Exercise 6.4 (The Quadratic form σ and Charge Conjugation)**.** We define charge conjugation on spinors in $V_+ \cong \mathbb{C}^2$ as:

$$\begin{aligned} J : V_+ &\to V_+ \\ \begin{pmatrix} a \\ b \end{pmatrix} &\mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \overline{a} \\ \overline{b} \end{pmatrix}. \end{aligned}$$

Prove that

$$\sigma(J\Phi, J\Phi) = -\sigma(\Phi, \Phi) \quad \forall \Phi \in V_+.$$

Problem Sheet 7

Curvature, Reducible Solutions, and Chern Classes

Exercise 7.1 (Non-negative Scalar Curvature)**.** Let (X, g) be a smooth closed oriented Riemannian 4-manifold with non-negative scalar curvature endowed with a Spin^c-structure $\mathfrak{s}$. Show that any solution (A, Φ) to the unperturbed Seiberg-Witten equations is reducible, that is, satisfies $\Phi = 0$.

Exercise 7.2 (Rescaling the Metric)**.** Let (X, g) be a smooth closed oriented Riemannian 4-manifold with a Spin^c-structure $\mathfrak{s}$ and let $\omega \in \Omega^2_+(X, i\mathbb{R})$. Consider the rescaled metric $\widetilde{g} = \lambda^2 g$ for $\lambda \in \mathbb{R}^+$.

(1) Prove that if (A, Φ) is a solution to the ω-perturbed Seiberg-Witten equations on (X, g), then $\left(A, \lambda^{-1}\Phi\right)$ satisfies the Seiberg-Witten equations on $(X, \widetilde{g})$.

[Notice that the Clifford module $\gamma : TX \to \mathrm{End}(V)$ is rescaled.]

(2) Show that the energy of the pair (A, Φ) with respect to the metric g equals the energy of $\left(A, \lambda^{-1}\Phi\right)$ with respect to the metric $\widetilde{g}$.

Exercise 7.3 (Reducible Solutions)**.** Let (X, g) be a smooth closed oriented Riemannian 4-manifold endowed with a Spin^c-structure $\mathfrak{s}$. Show that the unperturbed Seiberg-Witten equations admit a reducible solution if and only if the harmonic representative of $c_1\left(L_{\mathfrak{s}}\right)$ is anti-self-dual.

Exercise 7.4 (Chern Classes of Tensor Products, cf. Remark 4.41 in the Notes)**.**

(1) Let V, W be complex vector bundles of rank 2. Use the splitting principle to prove that

$$\begin{aligned} c_1(V \otimes W) &= 2\left(c_1(V) + c_1(W)\right) \\ c_2(V \otimes W) &= 2\left(c_2(V) + c_2(W)\right) + c_1^2(V) + c_1^2(W) + 3c_1(V)c_1(W). \end{aligned}$$

(2) Let V_+ be the spinor bundle of a Spin^c-structure over an oriented Riemannian 4-manifold. Use an isomorphism induced from Clifford multiplication to prove that

$$p_1\left(\Lambda^2_+\right) = c_1^2\left(V_+\right) - 4c_2\left(V_+\right).$$

Problem Sheet 8

Almost Complex Structures and Moduli Spaces

Exercise 8.1 (Almost Complex Manifolds I)**.** Show that a closed connected almost complex manifold X admits a canonical Spin^c structure $\mathfrak{s}$.

Exercise 8.2 (Almost Complex Manifolds II)**.** Let $\mathfrak{s}_L$ be the canonical Spin^c structure twisted by a line bundle L on a closed connected almost complex manifold X. Show that for generic parameters the expected dimension of the moduli space of irreducible solutions is

$$\dim\left(\mathcal{M}_\omega^*\right) = c_2\left((V_L)_+\right) = c_1^2(L) + c_1(L)c_1(X)$$

where $c_1(X)$ is the first Chern class of the complex vector bundle TX.

Exercise 8.3 (Reducible solutions II)**.** Let (X, g) be a smooth closed oriented Riemannian 4-manifold endowed with a Spin^c-structure $\mathfrak{s}$. Show that if $b_2^+(X) > 0$ and $c_1\left(L_{\mathfrak{s}}\right) \neq 0$, then for a generic metric on X the unperturbed Seiberg-Witten equations do not admit reducible solutions.

Exercise 8.4 (Theorem 5.16)**.** Show that the maps

$$\mathfrak{e} : \mathcal{G}^\perp \times S \longrightarrow \mathcal{C}_{\mathfrak{s}}$$

$$\left(e^{if}, (A_0 + a, \Phi)\right) \longmapsto \left(A_0 + a - i\,\mathrm{d}f, e^{if}\Phi\right)$$

and

$$\mathfrak{e}^{-1} : \mathcal{C}_{\mathfrak{s}} \longrightarrow \mathcal{G}^\perp \times S$$

$$(A_0 + b, \Psi) \longmapsto (e^{-G(\mathrm{d}^* b)}, A_0 + b - \mathrm{d}(G(\mathrm{d}^* b)), e^{G(\mathrm{d}^* b)}\Psi))$$

defined in Theorem 5.16 in the notes are inverse to each other.

Problem Sheet 9

Seiberg-Witten Equations on Canonical Manifolds

Exercise 9.1 (Seiberg-Witten Equations on the Flat Torus)**.** Consider $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with its flat Riemannian metric g_0 induced by the scalar product of $\mathbb{R}^4$. Prove the following statements.

(1) Any solution (A, Φ) to the unperturbed Seiberg-Witten equations on $\left(T^4, g_0\right)$ is reducible, i.e. Φ vanishes identically, and has flat $\hat{A}$.

(2) If the expected dimension of the moduli space for a Spin^c-structure $\mathfrak{s}$ on T^4 is non-negative, and the moduli space is non-empty, then the Spin^c-structure is the unique one induced by any spin structure (cf. Remark 2.32), and the moduli space is a copy of T^4.

Exercise 9.2 (The Even Expected Dimension Case)**.** Let (X, g) be a smooth closed oriented Riemannian 4-manifold endowed with a Spin^c-structure $\mathfrak{s}$. Show that if the expected dimension of the moduli space is even, then $b_2^+(X) - b_1(X)$ is odd.

Exercise 9.3 (Seiberg-Witten Equations on $S^2 \times S^2$)**.** Consider $S^2 \times S^2$ with the product metric, where each factor is a round sphere, i.e. of constant curvature.

(1) Determine the moduli spaces of solutions to the unperturbed Seiberg-Witten equations for all Spin^c-structures.

(2) Conclude that whenever the moduli space is non-empty, then the expected dimension is negative.

Exercise 9.4 (Seiberg-Witten Equations on $\#n\left(S^1 \times S^3\right)$)**.** Consider $S^1 \times S^3$, with the product metric coming from two round factors.

(1) Show that there is a unique Spin^c-structure, and determine the moduli space of solutions to the unperturbed Seiberg-Witten equations. How does the dimension of the result compare to the expected dimension?

(2) Extend this discussion to connected sums of several copies of $S^1 \times S^3$.

[Hint: you can use the fact that the connected sum of two Riemannian manifolds with positive scalar curvature admits a metric with positive scalar curvature.]

Problem Sheet 10

Perturbed Equations on Complex Projective Spaces

Exercise 10.1 (Metric of Positive Scalar Curvature on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$)**.** Prove that $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ has a metric with positive scalar curvature, without using the hint from Exercise 9.4.

Exercise 10.2 (Seiberg-Witten Equations on $\mathbb{CP}^2$)**.** Let $z = (z_1, z_2)$ be local coordinates on $\mathbb{CP}^2$. Consider the Fubini-Study metric

$$g_{FS} = \sum_{i,j=1}^{2} \frac{\delta_{ij}\left(1+|z|^2\right) - \bar{z}_i z_j}{\left(1+|z|^2\right)^2}\, \mathrm{d}z_i \otimes \mathrm{d}\bar{z}_j$$

associated to the Kähler form

$$\omega_{FS} = i\partial\bar{\partial}\log\left(1+|z|^2\right)$$

Note that ω_{FS} is a parallel 2-form.

(1) Classify Spin^c-structures on $\mathbb{CP}^2$ in terms of the cohomology.

(2) Show that the Fubini-Study metric has positive scalar curvature.

(3) Prove that for every Spin^c-structure the unperturbed Seiberg-Witten equation has no solution.

(4) Consider the perturbed Seiberg-Witten equations

$$\begin{aligned} D_A^+\Phi &= 0 \\ F_{\hat{A}}^+ &= \sigma(\Phi,\Phi) + i\varepsilon\omega_{FS} \end{aligned}$$

Show that for every Spin^c-structure there is a unique ε such that the equations have precisely one solution, which is reducible. What is the relation between this value of ε and the Spin^c-structure?

Exercise 10.3 (Small Perturbations of the Seiberg-Witten Equations on T^4)**.** Consider $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with its flat Riemannian metric g_0 induced by the scalar product of $\mathbb{R}^4$. Let $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$. Note that this is a parallel g_0-self-dual 2-form.

For a Spin^c-structure $\mathfrak{s} = (\gamma, V)$ on T^4 consider the perturbed Seiberg-Witten equations

$$\begin{aligned} D_A^+\Phi &= 0 \\ F_{\hat{A}}^+ &= \sigma(\Phi,\Phi) + i\varepsilon\omega \end{aligned}$$

where $0 < \varepsilon \ll 1$ is real and positive, and very small. Assume that the expected dimension of the moduli space of solutions is non-negative.

(1) Prove that if there is a solution to the equations, then $\left\langle c_1^2\left(L_{\mathfrak{s}}\right), \left[T^4\right]\right\rangle = 0$, equivalently the expected dimension is zero.

(2) For the unique Spin^c-structure with $c_1\left(L_{\mathfrak{s}}\right) = 0$ prove that there is precisely one solution up to gauge equivalence for every $\varepsilon \neq 0$.

Exercise 10.4 (Non-homotopic Almost Complex Structures)**.**

(1) Show that the space of orthogonal almost complex structures on $\mathbb{R}^4$ is $\mathrm{SO}(4)/\mathrm{U}(2) \cong \mathbb{CP}^1$.

(2) Show that there are almost complex structures J_0, J_1 on a suitable 4-manifold , e.g. $S^1 \times S^3$, that have the same Chern classes but that are not homotopic as almost complex structures.

[Hint: You can use the fact that $\pi_3\left(S^2\right) = \mathbb{Z}$.]

Problem Sheet 11

Integral Bounds and Spin Manifold Criteria

Exercise 11.1 (Unperturbed Seiberg-Witten Equation on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$)**.** Consider $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ endowed with a metric with positive scalar curvature as in Exercise 10.1.

(1) Classify Spin^c-structures on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ in terms of the cohomology.

(2) Compute the expected dimension of the moduli space of solutions to the unperturbed SeibergWitten equation for any Spin^c-structure.

(3) Prove that for every Spin^c-structure the unperturbed Seiberg-Witten equation has no solution.

Exercise 11.2 (Small Perturbations of the Seiberg-Witten Equations on T^4 II)**.** As in Exercise 10.3 consider $T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with its flat Riemannian metric g_0 induced by the scalar product of $\mathbb{R}^4$. Let $\omega = dx_1\wedge dx_2+dx_3\wedge dx_4$. For a Spin^c-structure $\mathfrak{s}$ on T^4 consider the perturbed Seiberg-Witten equations

$$\begin{aligned} D_A^+\Phi &= 0 \\ F_{\hat{A}}^+ &= \sigma(\Phi,\Phi) + i\varepsilon\omega \end{aligned}$$

where $0 < \varepsilon \ll 1$ is real and positive, and very small. Assume that the expected dimension of the moduli space of solutions is non-negative. Prove that $c_1(L_\mathfrak{s}) = 0$, as soon as there is a solution.

Exercise 11.3 (Some More Integral Bounds)**.** Let (X, g) be a closed, oriented, connected, Riemannian 4-manifold. For any solution (A,Φ) to the unperturbed Seiberg-Witten equation with non-negative expected dimension prove the following integral bounds:

$$\begin{aligned} \left\|F_{\hat{A}}^+\right\|_{L^2}^2 &\leqslant \frac{1}{8}\left\|s_{g,0}\right\|_{L^2}^2, \\ \left\|F_{\hat{A}}^-\right\|_{L^2}^2 &\leqslant \frac{1}{8}\left\|s_{g,0}\right\|_{L^2}^2 - 8\pi^2\chi(X) - 12\pi^2\sigma(X). \end{aligned}$$

What are the analogous bounds for parameters (g,ω) ?

Exercise 11.4 (Even Intersection Forms and Spin Manifolds)**.** Let X be a closed, oriented, connected, smooth 4-manifold without 2-torsion in $H^2(X;\mathbb{Z})$. Recall that a characteristic element for Q_X is an element $c \in H^2(X;\mathbb{Z})$ which satisfies

$$Q_X(c,a) \equiv Q_X(a,a) \mod 2$$

for all $a \in H^2(X;\mathbb{Z})$.

(1) Let $L_\mathfrak{s}$ be the characteristic line bundle of a Spin^c-structure $\mathfrak{s}$. Use the Atiyah Index Theorem to show that $c_1(L_\mathfrak{s})$ is a characteristic element for Q_X.

(2) Show that any characteristic element $c \in H^2(X;\mathbb{Z})$ satisfies $c = c_1(L_\mathfrak{s})$ for some Spin^c-structure $\mathfrak{s}$ on X.

(3) Conclude that a closed, oriented, connected, smooth 4-manifold X without 2-torsion in $H^2(X;\mathbb{Z})$ is spin if and only if its intersection form Q_X is even.

Problem Sheet 12

Diffeomorphisms and Invariant Computations

Exercise 12.1 (Complex Conjugation on $\mathbb{CP}^2$)**.** We want to show that there exists an orientation preserving diffeomorphism $d : \mathbb{CP}^2 \to \mathbb{CP}^2$ which is the identity on some ball $D^4 \subset \mathbb{CP}^2$ and induces $-\operatorname{Id}$ on $H_2\left(\mathbb{CP}^2; \mathbb{Z}\right)$.

(1) Consider the map $c : \mathbb{CP}^2 \to \mathbb{CP}^2$ given by complex conjugation of the homogeneous coordinates. Prove that c is orientation preserving and induces -Id on $H_2\left(\mathbb{CP}^2; \mathbb{Z}\right)$.

(2) Show that c preserves $\mathbb{C}^2 = \left\{[z_0 : z_1 : z_2] \in \mathbb{CP}^2 \mid z_0 = 1\right\}$ and find an explicit isotopy f_t on $\mathbb{C}^2$ with $f_0 = \operatorname{Id}_{\mathbb{C}^2}$ and $f_1 = c^{-1}\Big|_{\mathbb{C}^2}$.

(3) Let $D^4 \subset \mathbb{C}^2$ be a closed ball. Prove that c is isotopic to an orientation preserving diffeomorphism $d : \mathbb{CP}^2 \to \mathbb{CP}^2$ with $d\Big|_{D^4} = \operatorname{Id}_{D^4}$.

Exercise 12.2 (Reflection in (± 1)-sphere)**.**

(1) Let N be a smooth oriented 4-manifold and $M = N \# \mathbb{CP}^2$ or $M = N \# \overline{\mathbb{CP}}^2$. Let $E \in H_2(M; \mathbb{Z})$ be the homology class of the sphere $\mathbb{CP}^1 \subset \mathbb{CP}^2 \backslash D^4 \subset M$ with self-intersection $E^2 = \pm 1$. Use the diffeomorphism d from Exercise 12.1 to show that there exists an orientation preserving diffeomorphism $f : M \to M$ which induces on integer homology the map f_* given by

$$\begin{aligned} f_* : H_2(M; \mathbb{Z}) &\longrightarrow H_2(M; \mathbb{Z}) \\ A &\longmapsto A \mp 2(A \cdot E)E. \end{aligned}$$

(2) For an arbitrary smoothly embedded sphere S^2 of self-intersection ± 1 in a 4-manifold X, show that a tubular neighbourhood is diffeomorphic to a punctured $\mathbb{CP}^2$, and conclude that X is diffeomorphic to $Y \# \mathbb{CP}^2$ or $Y \# \overline{\mathbb{CP}}^2$, so that the above result is applicable.

Exercise 12.3 (Spheres with Self-intersection Zero)**.** Suppose that Y is a smooth closed oriented 4-manifold with $b_2^+(Y) \geqslant 2$ which contains a smoothly embedded $S^2 \hookrightarrow Y$ of self-intersection zero, representing a class of infinite order in $H_2(Y; \mathbb{Z})$. Consider the manifold $X = Y \# \overline{\mathbb{CP}}^2$.

(1) Prove that there exist infinitely many pairwise distinct classes $S_i \in H_2(X; \mathbb{Z})$ represented by embedded 2-spheres of self-intersection -1.

(2) Show that if there exists a Spin^c-structure $\mathfrak{s}$ on X with non-zero Seiberg-Witten invariant, then there are infinitely many such structures. Conclude that the Seiberg-Witten invariants of X and Y are in fact identically zero.

[Hint: You may use Theorem 8.31.]

Exercise 12.4 (Seiberg-Witten Invariants of $p\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$)**.** Let $X = p\mathbb{CP}^2 \# q\overline{\mathbb{CP}}^2$. Suppose that $p, q \geqslant 2$. Prove that $SW_X \equiv 0$.
[Hint: This appeared in Example 8.29, as a consequence of Theorem 8.28. It also follows from the fact that X admits metrics with positive scalar curvature. You should give a proof which is independent of these results, using Exercise 12.3 above.]