Ludwig-Maximilians-Universität München

Mathematical Gauge Theory II

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Contents

1 Introduction

1.1 The Seiberg-Witten Equations

The aim of this course is to discuss Seiberg-Witten theory on closed four-manifolds. This is the study of the solutions spaces for a set of partial differential equations, called the *Seiberg-Witten* or *monopole* equations. They are formulated on a smooth, closed, oriented 4-manifold equipped with a Spin^c-structure $\mathfrak{s}$, and look as follows:

$$D_A^+\Phi = 0\ ,$$
$$F_{\hat{A}}^+ = \sigma(\Phi,\Phi) + \omega\ .$$

These are equations for a pair (A, Φ), where

- Φ is a *positive (or left-handed) spinor*, i.e. a section of the (rank two) *spinor bundle* V_+ determined by $\mathfrak{s}$,
- $\hat{A}$ is a $U(1)$-connection on the line bundle $L = \Lambda^2 V_+$, induced by a Spin^c-connection A on V_+,
- D_A^+ is the *Dirac operator* on V_+ induced by A,
- $F_{\hat{A}}^+$ is the self-dual part of the curvature of $\hat{A}$, and
- $\sigma(\Phi,\Phi)$ is a self-dual 2-form defined by the spinor Φ.

The equations depend on the choice of a Spin^c-structure $\mathfrak{s}$ on the manifold, and, also on two further parameters:

(i) a Riemannian metric g on X, and

(ii) an imaginary-valued self-dual (with respect to g) 2-form ω on X.

Do not worry if you don't understand what all these words mean, since all but the most standard ones will be explained in the first few sections of this course. In fact, it will take us quite some time to set all this up precisely, before we can begin to study the solutions to the equations.

The Seiberg-Witten (SW) equations are *non-linear* PDEs, since, for example, $\sigma(\Phi,\Phi)$ is quadratic in Φ. To study the solution space of the SW equations for a given $(\mathfrak{s}, g, \omega)$ we use $\mathcal{C}_\mathfrak{s}$, the affine space of all configurations (A, Φ), and $\mathcal{Z}_\mathfrak{s} = \mathcal{Z}_\mathfrak{s}(g,\omega) \subset \mathcal{C}_\mathfrak{s}$, the space of all solutions to the SW equations.

The space of configurations is acted on by the *gauge group* $\mathcal{G} = C^\infty(X, S^1)$. This gives rise to the spaces $\mathcal{B}_\mathfrak{s} = \mathcal{C}_\mathfrak{s}/\mathcal{G}$ and $\mathcal{M}_\mathfrak{s} = \mathcal{M}_\mathfrak{s}(g,\omega) = \mathcal{Z}_\mathfrak{s}(g,\omega)/\mathcal{G}$. The latter is called the *Seiberg-Witten moduli space*. For a generic choice of (g,ω) this will turn out to be a smooth, finite-dimensional, closed, oriented manifold. Note that $\mathcal{B}_\mathfrak{s}$ is infinite-dimensional and that $\mathcal{M}_\mathfrak{s} \subset \mathcal{B}_\mathfrak{s}$. The *Seiberg-Witten invariants* are integrals of certain differential forms over the moduli space.

1.2 Physical Motivation

The study of the monopole equations was initiated by Seiberg and Witten in the context of supersymmetric field theory. In physics, a field theory on a manifold X is typically determined by a *Lagrangian* $\mathcal{L}$, which depends on A and Φ. Path integrals are integrals over functions on $\mathcal{C}_{\mathfrak{s}}$, weighted by $\exp(\int_X \mathcal{L})$.

In physics, configurations connected by a gauge transformation are regarded as equivalent (a principle called *gauge symmetry*) and therefore the path integral can be reduced to an integral over the quotient $\mathcal{B}_{\mathfrak{s}}$ (which is still infinite-dimensional). In the special field theory originally considered by Seiberg and Witten, there is a further reduction: In the limit of small coupling, the integral over $\mathcal{B}_{\mathfrak{s}}$ localizes to an integral over the (finite-dimensional) moduli space. Hence, the integrals are rigorously defined; they correspond to the SW invariants which we will construct.

It turns out that in many situations these path integrals do not depend on the choice of (g, ω). However, the SW invariants *can* depend on the smooth structure of X^4. Indeed, it is possible to construct 4-manifolds X and Y that are homeomorphic but admit different SW invariants, hence are not (orientation-preserving) diffeomorphic. This gives rise to a phenomenon called *exotic 4-manifolds*. During this course, we will proceed as follows:

(i) define and understand the structures appearing in the SW equations,

(ii) study the properties of the SW moduli space,

(iii) prove that the SW invariants often do not depend on choice of (g, ω), and

(iv) use these invariants to study the geometry and topology of smooth 4-manifolds.

2 Spin and Spin^c structures

In order to be able to define Dirac operators on manifolds, we first introduce spinors on Euclidean spaces through the notions of Clifford modules. Then we globalise this constructions on manifolds by introducing Spin and Spin^c structures. These structures will give rise to bundles on spinors, on which Dirac operators can be defined.

2.1 Clifford modules

Definition 2.1 (Clifford module)**.** Let H be a finite-dimensional, real vector space with a Euclidean scalar product g. A Clifford module for H is a finite dimensional complex vector space V equipped with a Hermitian scalar product together with a linear map $\gamma : H \to \mathrm{End}(V)$ satisfying

- $\gamma(v)^\dagger = -\gamma(v)$,
- $\gamma(v)\gamma(w) + \gamma(w)\gamma(v) = -2g(v, w)\mathrm{Id}_V$.

The elements of V are called *spinors* and the action $H \times V \to V$, $(v, \phi) \mapsto \gamma(v)\phi$ is called *Clifford multiplication*.

A Clifford module corresponds to a representation of the Clifford algebra $\mathrm{Cl}(H, g)$ of H on V, i.e. a homomorphism $\mathrm{Cl}(H, g) \to \mathrm{End}(V)$. A Clifford module is called *irreducible* if there are no nontrivial submodules. We will use the following result without proof:

Proposition 2.2. *If* $\dim H = 2m$ *then there is a unique (up to isomorphism), irreducible Clifford module* (V, γ) *with* $\dim V = 2^m$. *If* $\dim H = n = 2m + 1$ *then there exist two irreducible Clifford modules* (V, γ)*, and* $(V, -\gamma)$*, with* $\dim V = 2^m$.

We are interested mainly in the four-dimensional case, where $\dim_{\mathbb{R}} H = 4$ and $\dim_{\mathbb{C}} V = 4$. By the Proposition there is a unique irreducible Clifford module, which we describe explicitly below. By a *standard Clifford module* we will always mean one isomorphic to this one, so our development below does not in fact depend on the unproven Proposition.

Example 2.3 (Standard Clifford Module for $\mathbb{R}^4$)**.** Consider $\mathbb{R}^4$ with the standard Euclidean metric. For $V = \mathbb{C}^4$, we define γ as follows. Take the standard orthonormal basis $\{e_0, e_1, e_2, e_3\}$ for $\mathbb{R}^4$ and set

$$\gamma_0(e_j) = A_j := \begin{pmatrix} 0 & -B_j^\dagger \\ B_j & 0 \end{pmatrix} ,$$

where

$$B_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} ,\ B_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} ,\ B_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} ,\ B_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

To check that this defines a Clifford module for $(\mathbb{R}^4, g_{\text{std.}})$, we have to check that

- $\gamma_0(e_j)^\dagger = -\gamma_0(e_j)$
- $\gamma_0(e_i)\gamma_0(e_j) + \gamma_0(e_j)\gamma_0(e_i) = -2\delta_{ij}\mathrm{Id}_{\mathbb{C}^4}$.

This was the first exercise for the participants, Exercise 1 on Sheet 1. Here is the solution.

The first property holds trivially. To check the second, we begin by observing that

$$\gamma_0(e_i)\gamma_0(e_j) = -\begin{pmatrix} B_i^\dagger B_j & 0 \\ 0 & B_i B_j^\dagger \end{pmatrix} . \tag{2.1}$$

We now note that the B_i's satisfy $B_0^\dagger = B_0$, $B_k^\dagger = -B_k$ for $k = 1,\ 2,\ 3$. The latter follows from the fact that $B_1 = i\sigma_3$, $B_2 = i\sigma_1$, $B_3 = -i\sigma_2$, where σ_i are the Pauli matrices. The Pauli matrices satisfy the anti-commutation relations $\{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathrm{Id}_2$. Using equation (2.1), we find:

$$\{\gamma_0(e_i), \gamma_0(e_j)\} = -\begin{pmatrix} \{B_i, B_j\} & 0 \\ 0 & \{B_i, B_j\} \end{pmatrix} = -2\delta_{ij}\mathrm{Id}_4,$$

for $i,\ j \in \{1,\ 2,\ 3\}$. For $i = 0$, we have two cases: either $j = 0$ or $j \neq 0$. The first of these is trivial, and the second is also easy, using the same properties as before:

$$\{\gamma_0(e_0), \gamma_0(e_j)\} = -\begin{pmatrix} B_j + B_j^\dagger & 0 \\ 0 & B_j^\dagger + B_j \end{pmatrix} = 0\ .$$

Clifford multiplication $\mathbb{R}^4 \times \mathbb{C}^4 \to \mathbb{C}^4$ extends to the exterior algebra, i.e. Clifford multiplication by multivectors: $\Lambda^*\mathbb{R}^4 \times \mathbb{C}^4 \to \mathbb{C}^4$. For $i_1 < i_2 < \ldots < i_l$, we define

$$\gamma_0(e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_l}) = \gamma_0(e_{i_1})\gamma_0(e_{i_2}) \cdots \gamma_0(e_{i_l})\ .$$

Note that this definition does not work if two of the i_j's are the same, since $\gamma_0(e_i \wedge e_i) = \gamma_0(0) = 0$, but $\gamma_0(e_i)\gamma_0(e_i) = -\mathrm{Id}_V$. This means that $\Lambda^*H \ncong \mathrm{Cl}(H, g)$ as algebras even though $\Lambda^*H \cong \mathrm{Cl}(H, g)$ as vector spaces.

Lemma 2.4. *For the standard Clifford module for $\mathbb{R}^4$, we have*

$$\gamma_0(e_0 \wedge e_1 \wedge e_2 \wedge e_3) = \begin{pmatrix} -\mathrm{Id}_2 & 0 \\ 0 & \mathrm{Id}_2 \end{pmatrix}\ .$$

Hence we can split $\mathbb{C}^4 = \mathbb{C}^2_+ \oplus \mathbb{C}^2_-$ where the labels are *opposite* to the eigenspace decomposition of $\gamma_0(\mathrm{vol})$, i.e. $\gamma_0(\mathrm{vol})|_{\mathbb{C}^2_\pm} = \mp\mathrm{Id}_2$. The vector space $\mathbb{C}^4$ is called the space of *Dirac spinors*; $\mathbb{C}^2_+$ (respectively $\mathbb{C}^2_-$) is the space of *positive* or *left handed* (respectively *negative* or *right handed*) *Weyl spinors*.

Clifford multiplication has a nice relation to the Hodge star operator. Consider $(\mathbb{R}^n, g_{\mathrm{euclid}})$ with the standard volume form, i.e. $\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n$, for $\{\alpha_i\}$ the dual basis to $\{e_i\}$, an orthonormal basis.

Definition 2.5 (Hodge Duality on $\mathbb{R}^n$)**.** Hodge duality is a map $* : \Lambda^k(\mathbb{R}^n)^* \to \Lambda^{n-k}(\mathbb{R}^n)^*$, defined by

$$*(\alpha_{i_1} \wedge \alpha_{i_2} \wedge \ldots \wedge \alpha_{i_k}) = (-1)^\sigma(\alpha_{j_1} \wedge \alpha_{j_2} \wedge \ldots \wedge \alpha_{j_{n-k}})$$

Here, σ is the sign of the permutation that takes $(i_1\ i_2\ \ldots\ i_k\ j_1\ j_2\ \ldots\ j_{n-k})$ to $(1\ 2\ \ldots\ n)$.

Example 2.6 ($n = 4$, $k = 1$)**.**

$$\begin{aligned} *\alpha_1 &= \alpha_2 \wedge \alpha_3 \wedge \alpha_4\ , \\ *\alpha_2 &= -\alpha_1 \wedge \alpha_3 \wedge \alpha_4\ , \\ *\alpha_3 &= \alpha_1 \wedge \alpha_2 \wedge \alpha_4\ , \\ *\alpha_4 &= -\alpha_1 \wedge \alpha_2 \wedge \alpha_3\ . \end{aligned}$$

Lemma 2.7. $*^2 = \mathrm{Id}$ *on* $\Lambda^2(\mathbb{R}^4)^*$.

Proof. This simple computation is left as an exercise. $\square$

Since $*$ maps $\Lambda^2(\mathbb{R}^4)^*$ to itself, two-forms on $\mathbb{R}^4$ can be decomposed into a self-dual and anti self-dual part as follows:

$$\omega = \omega_+ + \omega_- = \frac{1}{2}(\omega + *\omega) + \frac{1}{2}(\omega - *\omega) \ .$$

In other words, there is an (orthogonal) decomposition $\Lambda^2(\mathbb{R}^4)^* = \Lambda^2_+(\mathbb{R}^4)^* \oplus \Lambda^2_-(\mathbb{R}^4)^*$. If $\{e_0, \dots e_3\}$ is an oriented, orthonormal basis of $\mathbb{R}^4$, we have the following basis for $\Lambda^2_\pm(R^4)$:

$$\begin{aligned} &e_0 \wedge e_1 \pm e_2 \wedge e_3 \ , \\ &e_0 \wedge e_2 \mp e_1 \wedge e_3 \ , \\ &e_0 \wedge e_3 \pm e_1 \wedge e_2 \ . \end{aligned}$$

Hence, $\dim \Lambda^2_\pm(\mathbb{R}^4)^* = 3$, and $\dim \Lambda^2(\mathbb{R}^4)^* = 6$. We end this section with two important observations regarding the standard Clifford module γ_0 for $\mathbb{R}^4$.

Lemma 2.8. *Under γ_0, the self dual 2-forms*

$$\begin{aligned} &e_0 \wedge e_1 + e_2 \wedge e_3 \\ &e_0 \wedge e_2 - e_1 \wedge e_3 \\ &e_0 \wedge e_3 + e_1 \wedge e_2 \end{aligned}$$

act non-trivially on $\mathbb{C}^2_+$ as $2B_1$, $2B_2$ and $2B_3$ respectively, and are zero on $\mathbb{C}^2_-$. An analogous result holds for $\Lambda^2_-(\mathbb{R}^4)^*$, with the roles of $\mathbb{C}^2_+$ and $\mathbb{C}^2_-$ interchanged.

Proof. This is another homework exercise, Exercise 2 on Sheet 1. $\square$

As a consequence, we have the following, also contained in Exercise 2 on Sheet 1.

Lemma 2.9. *γ_0 induces isomorphisms*

$$\begin{aligned} (\Lambda^1(\mathbb{R}^4) \oplus \Lambda^3(\mathbb{R}^4)) \otimes \mathbb{C} &\cong \mathrm{Hom}(\mathbb{C}^2_+, \mathbb{C}^2_-) \oplus \mathrm{Hom}(\mathbb{C}^2_-, \mathbb{C}^2_+) \ , \\ \Lambda^2_\pm(\mathbb{R}^4) \otimes \mathbb{C} &\cong \mathrm{End}_0(\mathbb{C}^2_\pm) \ , \\ \Lambda^4(\mathbb{R}^4) \otimes \mathbb{C} &\cong \mathbb{C} \cdot \mathrm{Id}_{\mathbb{C}^2_\pm} \ , \end{aligned}$$

where $\mathrm{End}_0(\mathbb{C}^2_\pm)$ is the space of trace-free endomorphisms of $\mathbb{C}^2_\pm$.

Since we said that a standard Clifford module for $\mathbb{R}^4$ will be one isomorphic to the example above, we need to make precise the notion of a Clifford module isomorphism.

Definition 2.10 (Clifford Module Isomorphism)**.** An isomorphism of Clifford modules (V, γ) and (V', γ') for H is a linear isometry $f : V \to V'$ such that $f \circ \gamma(v) = \gamma'(v) \circ f$ for all $v \in H$.

The following was Exercise 3 on Sheet 1:

Lemma 2.11 (Schur)**.** *Let (V, γ) be an irreducible Clifford module of H. Then every automorphism of V is of the form $f = \lambda \mathrm{Id}_V$ for a constant $\lambda \in S^1$.*

Proof. Since f is an isomorphism of Clifford modules it is an isometry, hence unitary. Therefore it can be diagonalized; each eigenvalue has unit norm. The requirement on f that it commute with γ means that γ preserves the eigenspaces of f:

$$f(\phi) = \lambda_i \phi \implies \gamma(f(\phi)) = f(\gamma(\phi)) = \lambda_i(\gamma(\phi)) \ .$$

Thus, each eigenspace is a submodule. Since γ is irreducible, V must be an eigenspace of f, i.e. $f = \lambda \mathrm{Id}$. Moreover, λ has unit norm, hence lies in S^1. □

2.2 Spin and Spin^c structures via Čech cohomology

2.2.1 Čech cohomology

Let X be a manifold, and S a *sheaf* over X. If G is a Lie group, S_G assigns to an open subset $U \subset X$ the continuous functions $U \to G$. We will mostly work with the Abelian groups S^1, $\mathbb{R}$ and $\mathbb{Z}$. For now, let S be any sheaf and consider a locally finite open covering $\mathcal{U} = \{U_a\}_{a \in A}$ of X. We define the *p-cochain groups* as the formal products

$$\begin{aligned} C^0(\mathcal{U}; S) &= \prod_a S(U_a) \ , \\ C^1(\mathcal{U}; S) &= \prod_{a \neq b} S(U_a \cap U_b) \ , \\ &\vdots \\ C^p(\mathcal{U}; S) &= \prod_{\substack{a_0, \ldots, a_p \\ \text{pairwise different}}} S(U_{a_0} \cap U_{a_1} \cap \ldots \cap U_{a_p}) \ . \end{aligned}$$

We now define the *coboundary* operator $\delta : C^p(\mathcal{U}; S) \to C^{p+1}(\mathcal{U}; S)$, given by

$$(\delta\sigma)_{a_0 \ldots a_{p+1}} := \prod_{j=0}^{p+1} \sigma^{(-1)^j}_{a_0 \ldots \hat{a}_j \ldots a_{p+1}} \Big|_{U_{a_0} \cap \ldots \cap U_{a_{p+1}}}$$

where the hat denotes omission.

We write out the coboundary operator for low values of p. If $\sigma = \{\sigma_a\} \in C^0$, then $(\delta\sigma)_{ab} = \sigma_b \sigma_a^{-1}|_{U_a \cap U_b}$. For $\sigma = \{\sigma_{ab}\} \in C^1$ we find $(\delta\sigma)_{abc} = \sigma_{bc}\sigma_{ac}^{-1}\sigma_{ab}|_{U_a \cap U_b \cap U_c}$, and so on.

Definition 2.12 (Čech Cohomology)**.** We call a p-cochain $\sigma \in C^p$ a *p-cocycle* if $\delta\sigma = 0$. The set of p-cocycles is denoted by $Z^p(\mathcal{U}; S)$. We say $\sigma \in C^p$ is a *p-coboundary*, i.e. an element of $B^p(\mathcal{U}; S)$ if $\sigma = \delta\tau$ for some $\tau \in C^{p-1}$. The Čech cohomology groups $\check{H}^p(X, S)$ are defined as the direct limit of $\check{H}^p(\mathcal{U}; S) := \frac{Z^p(\mathcal{U};S)}{B^p(\mathcal{U};S)}$, in the direct limit as the cover $\mathcal{U}$ gets finer.

Remark 2.13. On sufficiently nice spaces (e.g. manifolds), Čech cohomology of the sheaves of locally constant functions into $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Z}$ is isomorphic to singular cohomology with corresponding coefficients.

2.2.2 Spin^c structures

Let X be a manifold and $H \to X$ a real, oriented vector bundle equipped with a Euclidean bundle metric.

Definition 2.14 (Spin^c Structure)**.** A Spin^c-structure $\mathfrak{s}$ on $H \to X$ is a pair (V, γ) where $V \to X$ is a complex vector bundle with a Hermitian bundle metric, and γ a bundle homomorphism $\gamma : H \to \mathrm{End}(V)$ which is fiberwise a standard (irreducible) Clifford module.

Using Čech cohomology, we will find that the obstruction to the existence of Spin^c structures for $H \to X$ is a class $o(H) \in \check{H}^3(X; \mathbb{Z}) = H^3(X; \mathbb{Z})$. A Spin^c structure for H is constructed by by "gluing together" Spin^c-structures on the trivial bundles $H|_{U_a}$ (for a locally finite open covering $\{U_a\}_{a\in A}$ of X). More precisely, if we assume H has even rank then proposition 2.2 tells us that each $\mathfrak{s}_a$ is unique up to isomorphism. For each $U_{ab} = U_a \cap U_b$, choose an isomorphism $\phi_{ab} : \mathfrak{s}_a|_{U_{ab}} \to \mathfrak{s}_b|_{U_{ab}}$. Of course we can assume that $\phi_{aa} = \mathrm{Id}$ and $\phi_{ab} = \phi_{ba}^{-1}$.

This gluing procedure is (globally) consistent precisely if for every $U_{abc} = U_a \cap U_b \cap U_c$ we have $\phi_{ab}\phi_{bc} = \phi_{ac}$. Hence, the 2-cochain $\Psi_{abc} := \phi_{ab}\phi_{bc}\phi_{ca}$ is the identity for all $a,\ b,\ c$, if and only if the $\{\mathfrak{s}_a\}$ glue together to a Spin^c-structure for H. Since Ψ_{abc} is a fiberwise automorphism of $\mathfrak{s}_a|_{U_{abc}}$ we can apply Schur's lemma to view it as a collection of maps $\Psi_{abc} : U_{abc} \to S^1$ for all a, b, c pairwise disjoint. Hence, Ψ_{***} is a 2-cochain of S_{S^1}.

Lemma 2.15. *$\{\Psi_{***}\}$ is a 2-cocycle.*

Proof. This is just a computation:

$$(\delta\Psi)_{abcd} = \Psi_{bcd}\Psi_{acd}^{-1}\Psi_{abd}\Psi_{abc}^{-1} = \phi_{bc}\phi_{cd}\phi_{db}\phi_{ad}\phi_{dc}\phi_{ca}\phi_{ab}\phi_{bd}\phi_{da}\phi_{ac}\phi_{cb}\phi_{ba} = \mathrm{Id}|_{U_{abcd}}$$

because all the ϕ's cancel pairwise. □

Skipping the technicalities of direct limits, we obtain:

Theorem 2.16. *The cohomology class $o(H) := [\Psi_{***}] \in \check{H}^2(X; S_{S^1})$ is well-defined and independent of the choices of isomorphisms ϕ_{**} and covering $\mathcal{U}$.*

Proof. We will not prove independence of covering—this is a standard technical step that is taken care of in any discussion of sheaf cohomology. For independence of choices of isomorphisms, consider different isomorphisms $\phi'_{ab} = \pi_{ab}\phi_{ab}$ for $\pi_{ab} : U_{ab} \to S^1$. Then $\Phi'_{abc} = \pi_{ab}\pi_{bc}\pi_{ca}\Psi_{abc} = (\delta\pi)_{abc}\Psi_{abc}$ hence Ψ is modified only by a coboundary. □

The short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$, where the last map is the exponential map $t \mapsto \exp(2\pi i t)$, induces a long exact sequence on the level of sheaf cohomology:

$$0 \longrightarrow \check{H}^0(X, S_{\mathbb{Z}}) \longrightarrow \dots \longrightarrow \check{H}^p(X, S_{\mathbb{R}}) \longrightarrow \check{H}^p(X, S_{S^1}) \longrightarrow \check{H}^{p+1}(X, S_{\mathbb{Z}}) \longrightarrow \dots$$

Now, using the fact that the higher (meaning $p > 0$) sheaf cohomology groups of $S_{\mathbb{R}}$ vanish ($S_{\mathbb{R}}$ is a *fine* sheaf), we see by exactness that $\check{H}^p(X, S_{S^1}) \cong \check{H}^{p+1}(X, S_{\mathbb{Z}})$ for $p > 0$. Observing that continuous maps into $\mathbb{Z}$ are in fact locally constant, we have $\check{H}^p(X; S_{\mathbb{Z}}) = H^p(X; \mathbb{Z})$ and therefore may view $o(H)$ as a class in $H^3(X; \mathbb{Z})$.

Corollary 2.17. *A* Spin^c*-structure for* $H \to X$ *exists if and only if* $o(H) \in H^3(X; \mathbb{Z})$ *is trivial.*

We turn to the question of uniqueness. For $\mathfrak{s}$, $\mathfrak{s}'$ Spin^c-structures for H, we may assume (after potentially passing to a common refined covering) that we have a covering $\{U_a\}$ such that $H|_{U_a}$is trivial and we have Spin^c structures $\mathfrak{s}|_{U_a}$ and $\mathfrak{s}'|_{U_a}$. From the isomorphisms $\tau_a : \mathfrak{s}_a \to \mathfrak{s}'_a$, we construct automorphisms $\sigma_{ab} = \tau_a^{-1}\tau_b : U_{ab} \to S^1$.

Lemma 2.18.

(i) $\{\sigma_{**}\}$ *is a 1-cocycle.*

(ii) *The cohomology class* $\delta(\mathfrak{s}, \mathfrak{s}') := [\sigma_{**}] \in \check{H}^1(X, S_{S^1})$ *well defined.*

Proof.

(i) This follows directly from the definition.

(ii) Again, we do not check independence of covering. If we have other isomorphisms $\tau'_a = \mu_a\tau_a$ for $\mu_a : U_a \to S^1$, we obtain a cochain $\{\mu_*\}$, hence σ_{ab} changes only by the coboundary $\delta\mu$.

□

It is precisely $\delta(\mathfrak{s}, \mathfrak{s}')$ which is the obstruction to finding an isomorphism $\mathfrak{s} \to \mathfrak{s}'$. We can say more:

Lemma 2.19. *Given a* Spin^c*-structure* $\mathfrak{s}$ *and a class* $\alpha \in \check{H}^1(X; S_{S^1})$*, there exists some* $\mathfrak{s}'$ *with* $\delta(\mathfrak{s}, \mathfrak{s}') = \alpha$.

This is proven as follows. It is a standard fact from the theory of sheaf cohomology that $\check{H}^1(X, S_G)$ is the set of isomorphism classes of G-principal bundles. In the case of S^1, the identification $S^1 = U(1)$ tells us that this is also the set of isomorphism classes of complex line bundles (the transition maps into $\mathbb{C}^*$ can be taken to map into $U(1)$ after picking a Hermitian metric).

Recall that $\check{H}^1(X; S_{S^1}) \cong H^2(X;\mathbb{Z})$ and that for a fixed complex line bundle L, the first Chern class $c_1(L)$, which we will discuss in greater detail later on, is a class in $H^2(X;\mathbb{Z})$. In fact, the map $\check{H}^1(X; S_{U(1)}) \to H^2(X;\mathbb{Z})$ is (up to sign) given by c_1: The first Chern class classifies complex line bundles up to isomorphism. Hence, $\alpha \in \check{H}^1(X; S_{S^1})$ corresponds to a line bundle L_α with $c_1(L_\alpha) = \alpha$. Hence, $V_{\mathfrak{s}'} = V_{\mathfrak{s}} \otimes L_\alpha$ is a natural candidate for a Spin^c structure such that $\delta(\mathfrak{s}, \mathfrak{s}') = \alpha$. The proof is now finished by the following lemma:

Lemma 2.20. *The pair $(V_{\mathfrak{s}'}, \gamma_{\mathfrak{s}'})$ has a Clifford module structure, where $V_{\mathfrak{s}'} := V_{\mathfrak{s}} \otimes L_\delta$, $i : \mathrm{End}(V_{\mathfrak{s}}) \to \mathrm{End}(V_{\mathfrak{s}'})$ is an isomorphism, and $\gamma_{\mathfrak{s}'} = i \circ \gamma_{\mathfrak{s}}$.*

Proof. Left as an exercise. □

The above discussion may be summarized as follows:

Proposition 2.21. *Let $H \to X$ be a real, oriented vector bundle (i.e. associated to an $SO(n)$-principal bundle).*

(i) *The bundle H admits a Spin^c structure if and only if $o(H) \in H^3(X;\mathbb{Z})$ vanishes.*

(ii) *The set $\mathrm{Spin}^c(H)$, if non-empty, is an torsor for $H^2(X;\mathbb{Z})$ with the action of $H^2(X;\mathbb{Z})$ on $\mathrm{Spin}^c(H)$ given by $V_{\mathfrak{s}} \mapsto V_{\mathfrak{s}} \otimes L_\alpha$ for $\alpha \in H^2(X;\mathbb{Z})$.*

(A torsor is for a group what an affine space is for its vector space of translations.)

2.2.3 Spin structures

Spin structures are special kinds of Spin^c structures.

Definition 2.22 (Conjugate Vector Space)**.** Let V be a complex vector space. Then the conjugate vector space $\bar{V}$ is defined as follows:

- as an Abelian group, $V = \bar{V}$, and
- scalar multiplication is given by $\mathbb{C} \times V \to V$, $(\lambda, v) \mapsto \bar{\lambda}v$.

Remark 2.23. Note that $\mathrm{Id} : V \to \bar{V}$ is a complex antilinear map and that $\mathrm{End}\, V = \mathrm{End}\, \bar{V}$, hence if V yields a Spin^c structure for some H, then so does $\bar{V}$.

If H is even-dimensional, this implies that $V \cong \bar{V}$ as Clifford modules:

Lemma 2.24. *If H is even-dimensional with (V, γ) the unique irreducible Clifford module, then there exists a $\mathbb{C}$-linear map $J : V \to \bar{V}$ (i.e. a $\mathbb{C}$-antilinear map $V \to V$) that commutes with $\gamma(v)$ for all $v \in H$.*

Definition 2.25 (Charge Conjugation)**.** If $V \to X$ is a complex vector bundle, then there exists a complex conjugate vector bundle $\bar{V} \to X$. Since $\operatorname{End} V \cong \operatorname{End} \bar{V}$, we define charge conjugation as the map

$$\begin{aligned} \mathrm{Spin}^c(H) &\longrightarrow \mathrm{Spin}^c(H) \\ (V, \gamma) = \mathfrak{s} &\longmapsto \bar{\mathfrak{s}} = (\bar{V}, \gamma) \ . \end{aligned}$$

By the discussion in the previous section, the following definition always makes sense:

Definition 2.26 (Characteristic line bundle)**.** A complex line bundle $L_{\mathfrak{s}}$ such that $\mathfrak{s} = \bar{\mathfrak{s}} \otimes L_{\mathfrak{s}}$ (unique up to isomorphism) is called the characteristic line bundle of $\mathfrak{s}$.

Clearly, $\mathfrak{s} \cong \bar{\mathfrak{s}}$ if and only if $L_{\mathfrak{s}}$ is trivial. In this case, we say that $\mathfrak{s}$ arises from a Spin structure:

Definition 2.27 (Spin Structure)**.** A Spin^c-structure $\mathfrak{s}$ together with an isomorphism $J : \bar{\mathfrak{s}} \to \mathfrak{s}$ is called a Spin structure.

The following is clear:

Lemma 2.28. *The Spin^c-structure $\mathfrak{s}$ induced by a Spin structure for H is unique up to isomorphism.* □

We will consider the existence of Spin structures in section 2.3.2. On the question of uniqueness, we only mention the following result.

Lemma 2.29. *Suppose $(\mathfrak{s}, J)$ and $(\mathfrak{s}', J')$ are spin structures for $H \to X$. Then the line bundle L satisfying $\mathfrak{s} \cong \mathfrak{s}' \otimes L$ is 2-torsion, i.e. $2c_1(L) = 0 \in H^2(X; \mathbb{Z})$.*

Remark 2.30. This comes from the fact that the space of Spin structures is a torsor for the space of real line bundles (which are 2-torsion, since they are equal to their own dual). Therefore, two Spin^c structures induced by Spin structures differ by a complexified real line bundle (in some sense, this can be traced back to the fact that S^1 is to Spin^c structures what $\mathbb{Z}_2$ is to Spin structures).

Since the universal coefficient theorem tells us that $\operatorname{Tor} H^2(X; \mathbb{Z}) \cong \operatorname{Tor} H_1(X; \mathbb{Z})$, we obtain a nice corollary.

Corollary 2.31. *If $H_1(X; \mathbb{Z})$ is torsion-free, e.g. if X is simply connected, then two Spin structures $(\mathfrak{s}, J)$ and $(\mathfrak{s}', J')$ have to satisfy $\mathfrak{s} \cong \mathfrak{s}'$.*

Remark 2.32. Note that this is an isomorphism of the underlying Spin^c structures, not necessarily of Spin *structures*. In fact, T^4 is an example of a manifold with several Spin structures, but they all induce the same Spin^c structure. However, Spin structures are classified by $\check{H}^1(X;\mathbb{Z}_2)$ and therefore in case e.g. $\pi_1(X) = 1$, we have a(t most a) unique Spin structure.

2.3 The principal bundle point of view

2.3.1 Spin^c structures

In this section, we rephrase the existence of Spin^c structures using the formalism of principal bundles. We assume that n is even throughout. We denote by $\gamma_0 : \mathbb{R}^n \to \mathrm{End}\,\mathbb{C}^N$ the standard Clifford module.

Definition 2.33 ($\mathrm{Spin}^c(n)$)**.** The Lie group $\mathrm{Spin}^c(n)$ is defined as the set of pairs $(\tau, \sigma) \in \mathrm{SO}(n) \times \mathrm{U}(N)$ such that the following diagram commutes

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\ \tau\ } & \mathbb{R}^n \\ {\scriptstyle\gamma_0}\downarrow & & \downarrow{\scriptstyle\gamma_0} \\ \mathrm{End}\,\mathbb{C}^N & \xrightarrow{\mathrm{Ad}\,\sigma} & \mathrm{End}\,\mathbb{C}^N \end{array}$$

where $\mathrm{Ad}\,\sigma : \mathrm{End}\,\mathbb{C}^N \to \mathrm{End}\,\mathbb{C}^N$, $\mu \mapsto \sigma \circ \mu \circ \sigma^{-1}$ is the adjoint action.

Lemma 2.34. *The homomorphism* $\mathrm{Spin}^c(n) \to \mathrm{SO}(n)$*,* $(\tau, \sigma) \mapsto \tau$ *is surjective with kernel* $\{(\mathrm{Id}_n, S^1)\} \cong S^1$.

Proof. Let $\tau \in SO(n)$ be arbitrary. Then γ_0 and $\gamma_0 \circ \tau$ yield irreducible Clifford modules, hence are isomorphic (at least if n is even). Thus there must exist an isometry $\sigma \in U(n)$ such that $\sigma \circ \gamma_0(-) = \gamma_0 \circ \tau(-) \circ \sigma$. But then $\mathrm{Ad}\,\sigma \circ \gamma_0(-) = \gamma_0 \circ \tau(-)$, i.e. $(\tau, \sigma) \in \mathrm{Spin}^c(n)$.

For the kernel, assume (τ, σ) is mapped to $\mathrm{Id} \in \mathrm{SO}(n)$. Then clearly $\tau = \mathrm{Id}$ and we see that $\sigma \circ \gamma_0(-) \circ \sigma^{-1} = \gamma_0(-)$ hence σ commutes with γ_0, i.e. is an Clifford module isomorphism. Therefore it is given by an element $\lambda \in S^1$. (corresponding to a diagonal matrix with λ in the diagonal entries). □

Remark 2.35. One can show that $(\tau, \sigma) \in \mathrm{Spin}^c(n)$ if and only if $\sigma : \mathbb{C}^N \to \mathbb{C}^N$ is an isomorphism of Clifford modules covering the isometry $\tau : \mathbb{R}^n \to \mathbb{R}^n$.

Lemma 2.36. *Specifying a* Spin^c*-structure for an oriented Euclidean vector bundle* $H \to X$ *is equivalent to specifying a principal* $\mathrm{Spin}^c(n)$*-bundle* $Q \to X$ *together with an isomorphism* $Q/S^1 \cong \mathrm{Fr}(H)$*, where* $\mathrm{Fr}(H)$ *is the oriented, orthonormal frame bundle of* H.

Proof. Suppose we are given a Spin^c-structure $\mathfrak{s} = (V, \gamma)$, and want to define a $\mathrm{Spin}^c(n)$-bundle

$Q \to X$. We consider the sets of pairs of orientation-preserving linear isometries

$$(t, s) \in \mathrm{Isom}(\mathbb{R}^n, H_x) \times \mathrm{Isom}(\mathbb{C}^N, V_x) \cong \mathrm{SO}(n) \times \mathrm{U}(N)$$

that make the following diagram commute

$$\begin{array}{ccc} H_x & \xleftarrow{\ t\ } & \mathbb{R}^n \\ {\scriptstyle \gamma_x}\downarrow & & \downarrow{\scriptstyle \gamma_0} \\ \mathrm{End}(V_x) & \xleftarrow[\mathrm{Ad}(s)]{} & \mathrm{End}(\mathbb{C}^N) \end{array}$$

and set them equal to the fibers Q_x over $x \in X$. Then each fiber is diffeomorphic to $\mathrm{Spin}^c(n)$ by construction, and $Q_x/S^1 \ni [t, s] \mapsto t$ is a fiberwise isomorphism to $\mathrm{Fr}(H)$ by lemma 2.34. For the other direction, given a $\mathrm{Spin}^c(n)$-bundle $Q \to X$, we know we have a representation of $\mathrm{Spin}^c(n)$ on $\mathbb{C}^N$ given by $(\tau, \sigma) \mapsto \sigma \in \mathrm{U}(N)$. This yields an associated vector bundle $V \to X$ with fiber $\mathbb{C}^N$. It remains to show that the standard Clifford module γ_0 induces a Clifford module γ for V; this is left as an exercise. □

Since a G-principal bundle $P \to X$ is defined by a set $\{\gamma_{**}\}$ of transition functions, the class $[\gamma_{**}] := [P] \in H^1(X; S_G)$ corresponds to the isomorphism class of the bundle[1] P. From lemma 2.34 we obtain the short exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c(n) \xrightarrow{\ p\ } \mathrm{SO}(n) \longrightarrow 1$$

which induces a short exact sequence on the level of sheaves and thereby a long exact sequence[2]:

$$\cdots \longrightarrow \check{H}^1(X; S_{\mathrm{Spin}^c(n)}) \xrightarrow{\ p\ } \check{H}^1(X; S_{\mathrm{SO}(n)}) \xrightarrow{\ \delta\ } \check{H}^2(X; S_{S^1}) \longrightarrow \cdots$$

where p is the projection $(\tau, \sigma) \mapsto \tau$ and δ is the connecting homomorphism. Suppose now that the $\mathrm{SO}(n)$-bundle $\mathrm{Fr}(H)$ has isomorphism class $[H] = [\gamma_{**}] \in \check{H}^1(X; S_{\mathrm{SO}(n)})$; the above long exact sequence tells us that a $\mathrm{Spin}^c(n)$-bundle Q exists if there a lift of $[\gamma_{**}]$ to an element in $\check{H}^1(X; S_{\mathrm{Spin}^c(n)})$.

On the other hand, $\{\phi_{**}\}$ defined in section 2.2.2 is a 1-cochain of $S_{\mathrm{Spin}^c(n)}$ that covers $[\gamma_{**}]$ under p. If $[\phi_{**}]$ is a cocycle, we get a $\mathrm{Spin}^c(n)$-bundle Q such that $p[Q] = [H] \in \check{H}^1(X; S_{\mathrm{SO}(n)})$. It is clear from the definition of $[\Psi_{***}] = o(H) \in \check{H}^2(X; S_{S^1})$ that $[\phi_{**}]$ is a cocycle if and only if $\delta[H] = o(H) = 0 \in H^2(X; S_{S^1})$, in accordance with 2.17. This just means that we have a Spin^c structure precisely if $[H]$ lifts not just to a cochain but a co*cycle*.

We can also recover the obstruction to uniqueness (lemma 2.18): Suppose that $[Q]$, $[Q']$ are lifts of H. Then $p[Q] = p[Q']$, hence $p([Q] - [Q']) = 0$, so $[Q] - [Q']$ is in the image of $j : \check{H}^1(X; S_{S^1}) \to \check{H}^1(X; S_{\mathrm{Spin}^c(n)})$. We then find some $\delta(\mathfrak{s}, \mathfrak{s}') \in \check{H}^1(X; S_{S^1})$ such that $j(\delta(\mathfrak{s}, \mathfrak{s}')) = Q - Q'$, as before.

[1]This works even if G in non-Abelian. In this case, $\check{H}^1(X; S_G)$ is only a set, not a group. It has a special base point corresponding to the isomorphism class of the trivial bundle $[X \times G]$.

[2]Here, we are sweeping some technicalities under the rug: The spaces should simply be interpreted as pointed sets of isomorphism classes of G-bundles.

2.3.2 Spin structures and Stiefel-Whitney classes

Consider $(\mathbb{C}^N, \gamma_0, J_0)$, the standard Clifford module for $\mathbb{R}^n$ with its charge conjugation map.

Definition 2.37 ($\mathrm{Spin}(n)$)**.** Recall that $\mathrm{Spin}^c(n)$ was defined as a certain set of pairs $(\tau, \sigma) \in \mathrm{SO}(n) \times \mathrm{U}(N)$. We define $\mathrm{Spin}(n)$ as the subgroup of $\mathrm{Spin}^c(n)$ of elements (τ, σ) such that σ commutes with J_0.

Lemma 2.38. *The homomorphism* $\mathrm{Spin}(n) \to \mathrm{SO}(n)$*,* $(\tau, \sigma) \mapsto \tau$ *is surjective with kernel* $\{(\mathrm{Id}_n, \pm\mathrm{Id}_N)\} \cong \mathbb{Z}_2$*.*

Proof. This proof was left as Exercises 1 and 2 on Sheet 2. Before doing them, you should do Exercise 4 of Sheet 1. □

We therefore have the following short exact sequence:

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1 \,.$$

Comparing this with

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow \mathrm{SO}(n) \longrightarrow 1$$

we obtain the new description

$$\mathrm{Spin}^c(n) \cong (\mathrm{Spin}(n) \times S^1)/\mathbb{Z}_2 \,,$$

where $\mathbb{Z}_2$ identifies (τ, σ, λ) with $(\tau, -\sigma, -\lambda)$.

Lemma 2.39. *The map* $(\mathrm{Spin}(n) \times S^1)/\mathbb{Z}_2 \to \mathrm{Spin}^c(n)$*,* $[\tau, \sigma, \lambda] \mapsto (\tau, \lambda\sigma)$ *is an isomorphism.*

Proof. This is Exercise 3 on Sheet 2. □

Analogous to Spin^c-structures, which we saw are lifts of the frame bundle of a real, oriented vector bundle H to a $\mathrm{Spin}^c(n)$-bundle Q such that $Q/S^1 \cong \mathrm{Fr}\, H$, spin structures for H are lifts of $\mathrm{Fr}\, H$ to a $\mathrm{Spin}(n)$-bundle P such that $P/\mathbb{Z}_2 \cong \mathrm{Fr}(H)$. As before, we have a short exact sequence

$$0 \longrightarrow S_{\mathbb{Z}_2} \longrightarrow S_{\mathrm{Spin}(n)} \longrightarrow S_{\mathrm{SO}(n)} \longrightarrow 0$$

and hence a long exact sequence

$$\cdots \longrightarrow \check{H}^1(X; S_{\mathrm{Spin}(n)}) \xrightarrow{\ q\ } \check{H}^1(X; S_{\mathrm{SO}(n)}) \xrightarrow{\ \delta'\ } \check{H}^2(X; S_{\mathbb{Z}_2}) \cong H^2(X; \mathbb{Z}_2) \longrightarrow \cdots$$

and, arguing as before, we see that a lift $[P]$ for H exists if and only if $\delta'[H] = 0 \in H^2(X; \mathbb{Z}_2)$.

Definition 2.40 (Second Stiefel-Whitney Class)**.** Let H be a real, oriented vector bundle, and $\mathrm{Fr}\, H$ its $\mathrm{SO}(n)$-principal frame bundle. The second Stiefel-Whitney class $w_2(H)$ of H is defined as $w_2(H) = \delta'[H] \in H^2(X;\mathbb{Z}_2)$.

In terms of w_2, the above discussion can be summarized as:

Proposition 2.41. *A manifold is* Spin *if and only if it(s tangent bundle) has vanishing second Stiefel-Whitney class.*

Let us now examine the relation between $w_2(H)$ and the obstruction class for Spin^c-structures.

Lemma 2.42. *The obstruction class $o(H) \in \check{H}^2(X;S_{S^1})$ is the image of w_2 under $\iota^* : H^2(X;\mathbb{Z}_2) \to \check{H}^2(X;S_{S^1})$.*

Proof. The commutative ladder

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S^1 & \longrightarrow & \mathrm{Spin}^c(n) & \longrightarrow & \mathrm{SO}(n) & \longrightarrow & 1 \\ & & \iota\uparrow & & \uparrow & & \cong & & \\ 1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \mathrm{Spin}(n) & \longrightarrow & \mathrm{SO}(n) & \longrightarrow & 1 \end{array}$$

gives rise to

$$\begin{array}{ccccc} \check{H}^1(X, S_{\mathrm{Spin}^c(n)}) & \longrightarrow & \check{H}^1(X;S_{\mathrm{SO}(n)}) & \xrightarrow{\delta} & \check{H}^2(X;S_{S^1}) = H^3(X;\mathbb{Z}) \\ \uparrow & & \| & & \iota^*\uparrow \\ \check{H}^1(X;S_{\mathrm{Spin}(n)}) & \longrightarrow & \check{H}^1(X;S_{\mathrm{SO}(n)}) & \xrightarrow{\delta'} & H^2(X;\mathbb{Z}_2) \end{array}$$

Hence $\iota^*\delta'[H] = \delta[H] = o(H) = \iota^* w_2[H]$. ☐

Lemma 2.43. *The map $\iota : H^2(X;\mathbb{Z}_2) \to \check{H}^2(X;S_{S^1}) \cong H^3(X;\mathbb{Z})$ is equal to the* Bockstein homomorphism *$\beta : H^2(X;\mathbb{Z}_2) \to H^3(X;\mathbb{Z})$ induced by the short exact sequence*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\bmod 2} \mathbb{Z}_2 \longrightarrow 0$$

Proof. Consider the commutative ladder

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & S^1 & \longrightarrow & 0 \\ & & \| & & \uparrow\cdot\frac{1}{2} & & \iota\uparrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[\cdot 2]{} & \mathbb{Z} & \xrightarrow[\bmod 2]{} & \mathbb{Z}_2 & \longrightarrow & 0 \end{array}$$

Note that square on the right commutes since $\exp(2\pi i k/2) = (-1)^k$. On the level of cohomology we obtain

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & H^2(X; S_{S^1}) & \xrightarrow{\cong} & H^3(X;\mathbb{Z}) & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & & & \uparrow{\scriptstyle \iota^*} & & \| & & & & \\
\dots & \longrightarrow & H^2(X;\mathbb{Z}) & \xrightarrow{\text{mod } 2} & H^2(X;\mathbb{Z}_2) & \xrightarrow{\beta} & H^3(X;\mathbb{Z}) & \longrightarrow & H^3(X;\mathbb{Z}) & \longrightarrow & \dots
\end{array}$$

Under the identification $H^2(X; S_{S^1}) \cong H^3(X;\mathbb{Z})$, we see that $\iota^* = \beta$. □

The upshot of this analysis can be stated as follows.

Proposition 2.44. *The obstruction class $o(H) = \beta(w_2(H))$ vanishes if and only if $w_2(H)$ has a lift to $H^2(X;\mathbb{Z})$, that is, if $w_2(H)$ is in the image of $H^2(X;\mathbb{Z}) \xrightarrow{\text{mod } 2} H^2(X;\mathbb{Z}_2)$.*

Corollary 2.45. *A Spin^c-structure for H exists if and only if $w_2(H)$ is the $\bmod 2$ reduction of a class in $H^2(X;\mathbb{Z})$.*

2.4 $\mathrm{Spin}(4)$ and $\mathrm{Spin}^c(4)$

Recall that

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \;\middle|\; a,\ b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\} .$$

As a real vector space, we can view $\mathbb{R}^4$ as

$$\mathbb{R}^4 \cong \left\{ \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \in \mathrm{Mat}(2\times 2, \mathbb{C}) \;\middle|\; u,\ v \in \mathbb{C} \right\} \cong \mathbb{C}^2$$

and the action $\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathbb{R}^4 \to \mathbb{R}^4$, $(h_-, h_+, x) \mapsto h_- x h_+^{-1}$ defines a surjective homomorphism $\phi : \mathrm{SU}(2) \times \mathrm{SU}(2) \to \mathrm{SO}(4)$ with kernel $\{(\mathrm{Id}, \mathrm{Id}), (-\mathrm{Id}, -\mathrm{Id})\} \cong \mathbb{Z}_2$. Hence,

$$\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2) = \left\{ \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix} \;\middle|\; B_+,\ B_- \in SU(2) \right\} .$$

We remark that $\phi : \mathrm{Spin}(4) \to \mathrm{SO}(4)$ is the universal covering. To identify $\mathrm{Spin}^c(n)$, we recall $\mathrm{Spin}^c(n) \cong (\mathrm{Spin}(n) \times \mathrm{U}(1))/\mathbb{Z}_2$ and find

$$\begin{aligned}
\mathrm{Spin}^c(4) &= (\mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{U}(1))/((1,1,1) \sim (-1,-1,-1)) \\
&= \left\{ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} \;\middle|\; A_+, A_- \in \mathrm{SU}(2);\ \lambda \in \mathrm{U}(1) \right\} .
\end{aligned}$$

Let $H \to X$ be a real rank four Euclidean vector bundle equipped with a Spin^c-structure, which we regard as a Spin^c-bundle Q. Then we can construct the associated Weyl spinor bundles $S_\pm$, where the positive and negative spinors correspond (fiberwise) to $\mathbb{C}^4 = \mathbb{C}^2_+ \oplus \mathbb{C}^2_-$, as vector bundles associated to Q. The representations used to associate $S_\pm$ to Q are

$$\begin{aligned} \rho_\pm : \mathrm{Spin}^c(4) &\longrightarrow \mathrm{U}(2) \\ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} &\longmapsto \lambda A_\pm \end{aligned} .$$

Finally, from lemma 2.39, we recall that the map $(\mathrm{Spin}(n) \times S^1)/\mathbb{Z}_2 \to \mathrm{Spin}^c(n)$, $[\tau, \sigma, \lambda] \mapsto (\tau, \lambda\sigma)$ is an isomorphism. In particular, an element of $\mathrm{Spin}^c(n)$ unambiguously specifies a value of λ^2 and therefore the following representation is well-defined:

$$\begin{aligned} \chi : \mathrm{Spin}^c(4) &\longrightarrow \mathrm{U}(1) \\ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} &\longmapsto \lambda^2 \quad = \det(\lambda A_+) = \det(\lambda A_-) \end{aligned}$$

This representation associates the *determinant line bundle* $L \cong \det(S_+) = \Lambda^2(S_+) \cong \det(S_-) = \Lambda^2(S_-)$ to Q. The conjugate bundles are defined in the obvious way: $\bar{S} = \bar{S}_+ \oplus \bar{S}_-$, associated to $\mathfrak{s}$ via

$$\begin{aligned} \bar{\rho}_\pm : \mathrm{Spin}^c(4) &\to \mathrm{U}(2) \\ \begin{pmatrix} \lambda A_+ & 0 \\ 0 & \lambda A_- \end{pmatrix} &\mapsto \overline{\lambda A}_\pm \, . \end{aligned}$$

Lemma 2.46. *There exists a fixed matrix $M \in \mathrm{SU}(2)$ such that $MAM^\dagger = \bar{A}$ for all $A \in \mathrm{SU}(2)$.*

Proof. This was Exercise 4 on Sheet 2.

The Pauli matrices $\{\sigma_i\}$ form a basis of $\mathrm{SU}(2)$, hence one can work out the conditions imposed by the equations $M\sigma_i M^\dagger = \bar{\sigma}_i$. We obtain

$$M = \begin{pmatrix} 0 & \mp 1 \\ \pm 1 & 0 \end{pmatrix}$$

where we can pick either sign. □

This implies that the fundamental representation of $\mathrm{SU}(2)$ and its complex conjugate are isomorphic. Moreover,

$$\lambda^2 \overline{\lambda A}_\pm = \lambda |\lambda|^2 \bar{A}_\pm = \lambda \bar{A}_\pm = M(\bar{\lambda} A_\pm) M^\dagger$$

so the following representations are also isomorphic: $\rho_\pm \cong \chi \otimes \bar{\rho}_\pm$. Hence have $S_\pm \cong \bar{S}_\pm \otimes L_\mathfrak{s}$ and therefore $L_\mathfrak{s}$ is the characteristic line bundle.

If the spin structure changes from $\mathfrak{s}$ to $\mathfrak{s}'$, the determinant bundles are related in the following way.

Lemma 2.47. *Suppose $\mathfrak{s}' = \mathfrak{s} \otimes E$ for a complex line bundle E. Then $L_{\mathfrak{s}'} = L_\mathfrak{s} \otimes E^2$. Hence, $c_1(L_{\mathfrak{s}'}) = c_1(L_\mathfrak{s}) + 2c_1(E)$.*

Proof. We have that $\mathfrak{s} = \bar{\mathfrak{s}} \otimes L_\mathfrak{s}$, and $\mathfrak{s}' = \bar{\mathfrak{s}'} \otimes L_{\mathfrak{s}'}$. Then since $\mathfrak{s}' = \mathfrak{s} \otimes E$, we have that

$$\mathfrak{s} \otimes E = \bar{\mathfrak{s}} \otimes \bar{E} \otimes L_{\mathfrak{s}'} = \bar{\mathfrak{s}} \otimes L_\mathfrak{s} \otimes E \ .$$

This in turn implies that

$$L_\mathfrak{s} \otimes E = L_{\mathfrak{s}'} \otimes \bar{E} \implies L_\mathfrak{s} \otimes E^2 = L_{\mathfrak{s}'} \otimes \bar{E} \otimes E \ .$$

Now notice that $\bar{E} \otimes E = E^* \otimes E = \mathrm{End}(E)$ (the first equality follows because E is a line bundle); the latter bundle is in fact trivial for every line bundle E (a global section is given by Id_E). Hence, $L_{\mathfrak{s}'} = L_\mathfrak{s} \otimes E^2$. ☐

As a corollary, we see that $c_1(L_\mathfrak{s}) \mod 2 \in H^1(X; \mathbb{Z}_2)$ does not depend on the choice of $\mathfrak{s}$. We also note the following, without proof:

Proposition 2.48. $c_1(L_\mathfrak{s}) \equiv w_2(H) \mod 2$.

2.5 Spin^c-connections and Dirac operators

Let $H \to X$ be a real, oriented vector bundle equipped with a Euclidean metric g and a metric-compatible connection ∇^B. Assume H admits a Spin^c-structure $V \to X$ with Hermitian metric h (we will indicate Clifford multiplication by a dot).

Definition 2.49 (Spin^c-connection)**.** A Spin^c-connection A or ∇^A on V is a covariant derivative which is

(i) Hermitian, i.e. $L_Y h(\Phi, \Psi) = h(\nabla^A_Y \Phi, \Psi) + h(\Phi, \nabla^A_Y \Psi)$ for every $Y \in \mathfrak{X}(X)$ and $\Phi, \Psi \in \Gamma(V)$, and

(ii) compatible with ∇^B and Clifford multiplication in the sense that for every $Y \in \mathfrak{X}(X), \Phi \in \Gamma(V)$ and $T \in \Gamma(H)$ we have

$$\nabla^A_Y (T \cdot \Phi) = (\nabla^B_Y T) \cdot \Phi + T \cdot (\nabla^A_Y \Phi)$$

In case the first term vanishes, we have an obvious simplification:

Lemma 2.50. *Let ∇^A be a Spin^c-connection and T a parallel section of H with respect to ∇^B along the flow of a vector field Y. Then $\nabla^A_Y (T \cdot \Phi) = T \cdot \nabla^A_Y \Phi$.* ☐

This is sometimes useful when verifying identities pointwise (where one may choose a local frame of parallel sections).

2.5.1 The Dirac operator on $\mathbb{R}^n$

Loosely speaking, the Dirac operator is the "square root" of the Laplace operator. Let us try to formalize this idea. For a function $\phi : \mathbb{R}^n \to \mathbb{C}^N$, the Dirac operator D can be written in a basis as

$$D\Phi = \sum_{i=1}^{n} A_i \frac{\partial \Phi}{\partial x_i}$$

where the A_i's are constant, complex $N \times N$ matrices. The Laplace operator on $\mathbb{R}^n$ is given by

$$\Delta\Phi = -\sum_{i=1}^{n} \mathrm{Id}_N \frac{\partial^2 \Phi}{\partial x_i^2}$$

Imposing $D^2 = \Delta$, we obtain

$$\begin{aligned} D^2\Phi &= \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j} D\Phi = \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j} \sum_{i=1}^{n} A_i \frac{\partial \Phi}{\partial x_i} \\ &= \sum_{i,j=1}^{n} A_j A_i \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \overset{!}{=} -\sum_{i=1}^{n} \mathrm{Id}_N \frac{\partial^2 \Phi}{\partial x_i^2} \end{aligned}$$

which is equivalent to

$$A_i^2 = -\mathrm{Id}_N \qquad \text{and} \qquad A_j A_i + A_i A_j = 0 \quad \forall\, i \neq j\ . \tag{2.2}$$

We also want D to be formally self-adjoint. For $\Phi,\ \Psi : \mathbb{R}^n \to \mathbb{C}^n$, we consider the L^2-scalar product

$$\langle \Phi, \Psi \rangle = \int_{\mathbb{R}^n} \mathrm{d}^n x \Phi^\dagger \Psi\ .$$

Now, we require $\langle D\Phi, \Psi \rangle = \langle \Phi, D\Psi \rangle$:

Lemma 2.51. *D is formally self-adjoint if and only if $A_i^\dagger = -A_i$ (i.e. A_i is skew-adjoint).*

Proof. This is a simple computation. On the one hand

$$\langle D\Phi, \Psi \rangle = \int_{\mathbb{R}^n} \mathrm{d}^n x \sum_i \frac{\partial \Phi^\dagger}{\partial x_i} A_i^\dagger \Psi\ ,$$

while on the other hand, integration by parts shows that

$$\langle \Phi, D\Psi \rangle = \int_{\mathbb{R}^n} \mathrm{d}^n x \Phi^\dagger \sum_i A_i \frac{\partial \Psi^\dagger}{\partial x_i} = -\int_{\mathbb{R}^n} \mathrm{d}^n x \sum_{i=1}^{n} \frac{\partial \Phi^\dagger}{\partial x_i} A_i \Psi\ .$$

Hence, $\langle D\Phi, \Psi \rangle = \langle \Phi D\Psi \rangle$ if and only if $A_i^\dagger = -A_i$. □

Equation (2.2) together with the above lemma indicate that $A_i^\dagger A_i = \mathrm{Id}_N$ for each i, i.e. the A_i's are unitary.

2.5.2 The Dirac operator on a spinor bundle

We specialize to the following set-up. Let $H = TX \to X$, where X is an oriented manifold, equipped with a Riemannian metric g. Let $\nabla^B = \nabla$, be the unique torsion-free[3], metric-compatible connection (called the Levi-Civita connection). Let $\mathfrak{s}$ be a Spin^c-structure on X.

Definition 2.52 (Dirac Operator)**.** If ∇^A is a Spin^c-connection on V, the Dirac operator D_A : $\Gamma(V) \to \Gamma(V)$ is defined as the composition

$$D_A : \Gamma(V) \xrightarrow{\nabla^A} \Gamma(T^*X \otimes V) \xrightarrow{g} \Gamma(TX \otimes V) \xrightarrow{\gamma_{\text{eval}}} \Gamma(V)$$

where γ_{eval} denotes γ composed with Clifford multiplication (evaluation).

Lemma 2.53. *Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of (T_pX, g_p). Then*

$$D_A\Phi = \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}\Phi \ .$$

Remark 2.54. In physics, one typically writes $D\Phi = i\gamma^\mu\partial_\mu\Phi$. Clearly, γ^μ corresponds to Clifford multiplication by e_μ while ∂_μ corresponds to $\nabla^A_{e_\mu}$. The factor i arises from the different convention $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$.

Proof of Lemma. Let $\{\omega_i\}$ be the dual basis to $\{e_i\}$. Then the covariant derivative in coordinates is given by

$$\nabla^A_Y\Phi = \sum_{i=1}^{n} Y_i\nabla^A_{e_i}\Phi = \sum_i \omega_i(Y)\nabla^A_{e_i}\Phi = \Big(\sum_i \omega_i \otimes \nabla^A_{e_i}\Phi\Big)(Y) \ .$$

Hence, the Dirac operator is given by

$$\begin{aligned} D_A\Phi &= \gamma_{\text{eval}} \circ g \circ \nabla^A\Phi = \gamma_{\text{eval}} \circ g \left(\sum_{i=1}^{n} \omega_i \otimes \nabla^A_{e_i}\Phi\right) = \gamma_{\text{eval}} \left(\sum_{i=1}^{n} e_i \otimes \nabla^A_{e_i}\Phi\right) \\ &= \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}\Phi \ . \end{aligned}$$

□

Definition 2.55 (L^2 scalar product on spinors)**.** In the above setting, suppose (X, g) is closed. We define a Hermitian scalar product on the space of smooth sections $\Gamma(V)$ given by

$$\langle\Phi, \Psi\rangle = \int_X h(\Phi, \Psi)\mathrm{vol}_g$$

where h is the Hermitian scalar product on the spinor bundle.

[3] Recall that this condition means that for $Y, Z \in \mathfrak{X}(X)$, $\nabla_Y Z - \nabla_Z Y = [Y, Z]$.

Proposition 2.56. *With respect to the L^2 scalar product, the Dirac operator D_A is formally self-adjoint.*

The proof of this proposition relies on the following result:

Lemma 2.57. *For η the 1-form defined by $\eta(X) = h(X \cdot \Phi, \Psi)$, the following holds:*

$$h(D_A\Phi, \Psi) - h(\Phi, D_A\Psi) = * \,\mathrm{d} * \eta \,.$$

Proof. We prove this pointwise. Let (e_j) be a local frame of V around $p \in X$ such that $(\nabla e_i)_p = 0$, i.e. the e_i are parallel in p. Using that ∇^A is a Spin^c connection, we have (in p):

$$\begin{aligned} h(D_A\Phi, \Psi) - h(\Phi, D_A\Psi) &= \sum_i \Big(h\big(\nabla_{e_i}(e_i \cdot \Phi), \Psi\big) - h\big(\Phi, e_i \cdot \nabla^A_{e_i}\Psi\big)\Big) \\ &= \sum_i \Big(h\big(\nabla_{e_i}(e_i \cdot \Phi), \Psi\big) + h\big(e_i \cdot \Phi, \nabla^A_{e_i}\Psi\big)\Big) \\ &= \sum_i L_{e_i} h(e_i \cdot \Phi, \Psi) = \sum_i L_{e_i}\eta(e_i) = * \,\mathrm{d} * \eta \end{aligned}$$

where we used that Clifford multiplication is skew-Hermitian to pass to the second line. □

Proof of Proposition. This is now an easy application of Stokes's theorem:

$$\langle D_A\Phi, \Psi\rangle - \langle D_A\Phi, \Psi\rangle = \int_X * \,\mathrm{d} * \eta \,\mathrm{vol}_g = \int_X \mathrm{d} * \eta = 0 \,.$$

□

Example 2.58 (D_A on X^4)**.** Let us consider the case where X^4 is an oriented 4-manifold, $H = TX$ and $V = V_+ \oplus V_-$ the Clifford module defined by a Spin^c structure on X.

Lemma 2.59. *Every Spin^c-connection on V preserves $V_\pm$.*

Proof. The volume form is always parallel with respect to the Levi-Civita connection ∇. Recall that $V_\pm$ are the $\mp$-eigenspaces of V under Clifford multiplication by vol_g. Because vol_g is parallel, a Spin^c-connection must commute with this Clifford multiplication. Thus, we obtain

$$\mp \nabla^A_Y \Phi_\pm = \nabla^A_Y(\mathrm{vol}_g \cdot \Phi_\pm) = \mathrm{vol}_g \cdot \nabla^A_Y \Phi_\pm$$

hence $\nabla^A_Y \Phi_\pm \in \Gamma(V_\pm)$. □

Lemma 2.60. *Clifford multiplication with a tangent vector exchanges V_+ and V_-*

Proof. This follows from the exercise where we showed $(\Lambda^1\mathbb{R}^4 \oplus \Lambda^3\mathbb{R}^4) \otimes \mathbb{C} \cong \mathrm{End}(\mathbb{C}^2_+, \mathbb{C}^2_-) \oplus \mathrm{End}(\mathbb{C}^2_-, \mathbb{C}^2_+)$. □

Therefore, the Dirac operator decomposes as

$$D_A\Phi = \begin{pmatrix} 0 & D_A^- \\ D_A^+ & 0 \end{pmatrix} \begin{pmatrix} \Phi_+ \\ \Phi_- \end{pmatrix}$$

where $D_A^{\pm} : \Gamma(V_{\pm}) \to \Gamma(V_{\mp})$.

2.5.3 The existence of Spin^c-connections

In this section, we think of Spin^c-structures as $\mathrm{Spin}^c(n)$-principal bundles. Consider the $\mathrm{SO}(n)$-principal bundle $\mathrm{Fr}\, H$ and recall the bijective correspondence between connection 1-forms on a principal bundle and covariant derivatives on associated bundles. By this correspondence, the covariant derivative ∇^B defines a principal connection B on $\mathrm{Fr}(H)$, i.e. $B \in \Omega^1(\mathrm{Fr}(H), \mathfrak{so}(n)) = \Gamma(T^* \mathrm{Fr}(H) \otimes \mathfrak{so}(n))$ such that

(i) $r_g^* B = \mathrm{Ad}(g^{-1}) \circ B$ for every $g \in \mathrm{SO}(n)$, where r_g is right-multiplication by g, and

(ii) $B(\widetilde{W}) = W$ for all $W \in \mathfrak{so}(n)$, $\tilde{W}$ the fundamental vector field defined by $W \in \mathfrak{so}(n)$.

Now let $A \in \Omega^1(Q, \mathfrak{spin}^c(n))$ be a connection on the $\mathrm{Spin}^c(n)$-bundle $Q \to X$; it defines a Hermitian covariant derivative ∇^A on $V \to X$, where $V \to X$ is the vector bundle associated to $Q \to X$ via the representation

$$\begin{aligned} \rho : \mathrm{Spin}^c(n) &\longrightarrow \mathrm{U}(N) \\ (\tau, \sigma) &\longmapsto \sigma \end{aligned} .$$

Let $\varrho : \mathrm{Spin}^c(n) \to \mathrm{SO}(n)$, $(\tau, \sigma) \mapsto \tau$ be the surjective homomorphism defined in lemma 2.34. We denote the induced Lie algebra homomorphisms by ρ_* and ϱ_*.

We wish to answer the following question: In the formalism of principal bundles, what does is it mean for A that ∇^A is a Spin^c-connection? Which conditions are imposed on A? We have the following results.

Lemma 2.61. *For $Y \in \mathfrak{spin}^c(n)$,*

$$\varrho_*(Y)(t \cdot \phi) = \rho_*(Y)t \cdot \phi + t \cdot \varrho_*(Y)\phi$$

for all $t \in \mathbb{R}^n$, $\phi \in \mathbb{C}^N$.

Proof. By the definition of $\mathrm{Spin}^c(n)$, $g \in \mathrm{Spin}^c(n)$ satisfies the equation $\rho(g)(t \cdot \phi) = (\varrho(g)(t)) \cdot (\rho(g)\phi)$. Differentiating this, we obtain the result we are looking for. □

Proposition 2.62. *Let $\pi : Q \to \operatorname{Fr} H$ be the bundle map that identifies $Q/S^1 \cong \operatorname{Fr} H$. A Hermitian covariant derivative ∇^A is a Spin^c-connection if and only if the following diagram commutes:*

$$\begin{array}{ccc} TQ & \xrightarrow{A} & \mathfrak{spin}^c(n) \\ {\scriptstyle D\pi}\big\downarrow & & \big\downarrow{\scriptstyle \varrho_*} \\ T\operatorname{Fr}(H) & \xrightarrow[B]{} & \mathfrak{so}(n) \end{array} .$$

Proof. Let $s : U \to Q$ be a local section on an open set $U \subset X$. It induces a section $\pi \circ s$ of $\operatorname{Fr} H$. We have the associated bundles $V = Q \times_\rho \mathbb{C}^N$ and $H = \operatorname{Fr} H \times_{\text{std}} \mathbb{R}^n$. Let $\Phi = [s, \phi] \in \Gamma(V, U)$, $T = [t, \pi \circ s] \in \Gamma(H, U)$ and $Y \in \mathfrak{X}(X)$. We have the covariant derivatives

$$\nabla_Y^A \Phi = [s, L_Y \phi + \rho_*(s^* A(Y))\phi] \qquad \nabla_Y^B T = [\pi \circ s, L_Y t + ((\pi \circ s)^* B(Y))t]$$

omitting the $[s, \dots]$ and $[\pi \circ s, \dots]$ for simplicity, we have:

$$\begin{aligned} \nabla_Y^A(T \cdot \phi) &= L_Y(t \cdot \phi) + \rho_*(s^* A(Y))(t \cdot \phi) \\ (\nabla_Y^B T) \cdot \phi &= (L_Y t) \cdot \phi + ((\pi \circ s)^* B(Y) t) \cdot \phi \\ T \cdot \nabla_Y^A \phi &= t \cdot L_Y \phi + t \cdot \rho_*(s^* A(Y))\phi \,. \end{aligned}$$

Now ∇^A is a Spin^c-connection precisely if $\nabla_Y^A(T \cdot \Phi) + T \cdot \nabla_Y^A \Phi$. Setting $s^* A(Y) = a$, it is clear that all we need to show is

$$\rho_*(a)(t \cdot \phi) = ((\pi \circ s)^* B(Y) t) \cdot \phi + t \cdot \rho_*(a)\phi \,.$$

Using the previous lemma, this is equivalent to requiring that $(\pi \circ s)^* B(Y) = \varrho_*(a)$. But this means exactly that $s^* \circ \pi^* B = B \circ D\pi \circ Ds = \varrho_* \circ A \circ Ds$, i.e. $B \circ D\pi = \varrho_* \circ A$, at least on the image of Ds, but that is all we need for it to be true for the covariant derivative. $\square$

Corollary 2.63. ∇^A *is a* Spin^c*-connection compatible with* ∇^B *if and only if* ∇^B *is associated to the principal connection A on Q via the representation $\varrho : \operatorname{Spin}^c(n) \to \operatorname{SO}(n)$. More explicitly, for $T = [\pi \circ s, t] \in \Gamma(H)$, where $s : U \subset X \to Q$ is a local section and $t : U \to \mathbb{R}^n$, we can write*

$$\nabla_Y^B T = [\pi \circ s, L_Y t + \varrho_*(s^* A(Y))t] \,.$$

Let $L \coloneqq L_{\mathfrak{s}}$ denote the characteristic line bundle associated to Q via the homomorphism

$$\begin{aligned} \chi : \operatorname{Spin}^c(n) &\to \operatorname{U}(1) \\ [\tau, \sigma, \lambda] &\mapsto \lambda^2 \,. \end{aligned}$$

Then $\chi_* : \mathfrak{spin}^c(n) \to \mathfrak{u}(1)$ is a Lie algebra homomorphism. We further define $\mathcal{L}$ as the $\operatorname{U}(1)$-principal bundle corresponding to L, i.e. $\mathcal{L}$ is the complex frame bundle $\operatorname{Fr}^{\mathbb{C}}(L)$ of L.

Proposition 2.64. *A connection A on Q induces a connection $\mathcal{A}$ on $\mathcal{L}$ such that the following diagram commutes:*

$$\begin{array}{ccc} TQ & \xrightarrow{A} & \mathfrak{spin}^c(n) \\ {\scriptstyle Dc}\big\downarrow & & \big\downarrow{\scriptstyle \chi_*} \\ T\mathcal{L} & \xrightarrow[\mathcal{A}]{} & \mathfrak{u}(1) \end{array} .$$

Here $c : Q \to \mathcal{L}$ is the bundle map induced by the representation χ,

Since $\mathrm{Spin}^c(n) \cong \mathrm{Spin}(n) \times U(1)/\mathbb{Z}_2$, the Lie algebra is given by $\mathfrak{spin}^c(n) \cong \mathfrak{spin}(n) \oplus \mathfrak{u}(1) \cong \mathfrak{so}(n) \oplus \mathfrak{u}(1)$. The isomorphism $\mathfrak{spin}^c(n) \to \mathfrak{so}(n) \oplus \mathfrak{u}(1)$ is given by $(\rho \times \chi)_*$. This allows us to construct a connection on Q from connections on $\mathrm{Fr}\, H$ and $\mathcal{L}$, bringing us to the main result of this section:

Theorem 2.65. *Let B be a principal $\mathrm{SO}(n)$-connection on $\mathrm{Fr}(H)$, and $\mathcal{A}$ a principal $\mathrm{U}(1)$-connection on $\mathcal{L}$. Then there exists a unique principal $\mathrm{Spin}^c(n)$-connection A on Q such that the following diagrams commute:*

$$\begin{array}{ccc} TQ & \xrightarrow{A} & \mathfrak{spin}^c(n) \\ {\scriptstyle D\pi}\big\downarrow & & \big\downarrow{\scriptstyle \varrho_*} \\ T\,\mathrm{Fr}(H) & \xrightarrow{B} & \mathfrak{so}(n) \end{array} \qquad \begin{array}{ccc} TQ & \xrightarrow{A} & \mathfrak{spin}^c(n) \\ {\scriptstyle Dc}\big\downarrow & & \big\downarrow{\scriptstyle \chi_*} \\ T\mathcal{L} & \xrightarrow{\mathcal{A}} & \mathfrak{u}(1) \end{array} .$$

Proof. Let us denote the $SO(n) \times U(1)$-bundle on X with fiber $\mathrm{Fr}\, H_x \times \mathcal{L}_x$ by $\mathrm{Fr}\, H \tilde{\times} \mathcal{L}$. Consider a connection $B \oplus \mathcal{A}$ on this bundle and the bundle map $\pi \tilde{\times} c : Q \to \mathrm{Fr}\, H \tilde{\times} \mathcal{L}$. This is in fact a 2:1 covering, but we only need it to be a local diffeomorphism. This is guaranteed by the fact that we have an isomorphism $(\rho \times \chi)_*$, which implies that $D(\pi \tilde{\times} c)$ is an isomorphism as a map between tangent spaces between each point, i.e. on the fibers of the tangent bundles. This allows us to define A to be the map that makes the following diagram commute:

$$\begin{array}{ccc} TQ & \xrightarrow{\quad A \quad} & \mathfrak{spin}^c(n) \\ {\scriptstyle \text{iso on fibers}}\big\downarrow{\scriptstyle D(\pi\tilde{\times}c)} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle (\rho\times\chi)_*} \\ T(\mathrm{Fr}\, H \tilde{\times} \mathcal{L}) & \xrightarrow{B\oplus\mathcal{A}} & \mathfrak{so}(n) \oplus \mathfrak{u}(1) \end{array}$$

This uniquely defines A because the vertical maps are isomorphisms (on fibers). □

Corollary 2.66. *The choice of a metric connection ∇^B on H and a Hermitian connection $\nabla^{\mathcal{A}}$ on $\mathcal{L}$ determines a unique Spin^c-connection ∇^A on V.*

3 Classification of manifolds in low dimensions

Since we want to apply Seiberg-Witten gauge theory to study the geometry and topology of four-dimensional manifolds, we now give a quick overview of the classification of manifolds in small dimensions, followed by an introduction to some of the classical topics in four-dimensional topology.

The manifolds we consider are (almost always) smooth, and most of the time they are connected, closed, and oriented. Here the word *closed* is an abbreviation for compact and without boundary.

3.1 Dimensions up to three

3.1.1 Dimension 1

In dimension 1 the classification is quite elementary. Indeed, it is not difficult to prove that a 1-dimensional manifold M that is closed and connected is diffeomorphic to the 1-dimensional sphere S^1. One uses compactness to patch together the manifold from finitely many intervals which make up the charts of a finite atlas.

3.1.2 Dimension 2

In dimension 2 the classification is a bit more interesting, although it has been understood completely for more than 100 years. Surfaces are classified, up to diffeomorphism, by their genus. For genus 0 we have the sphere S^2, for genus 1 the torus T^2, then the surface Σ_2 of genus 2, the surface Σ_3 of genus 3, and so on. The genus can be defined in several different ways. For example, it equals half the first Betti number of M: $g(M) = \frac{1}{2}\,\mathrm{b}_1(M) = \frac{1}{2}\dim H^1(M;\mathbb{R})$. Note that implicit in this definition is the non-trivial statement that the first Betti number of M is necessarily even. We can think of the genus as the number of "holes" of a surface, but this intuition only applies to surfaces embedded in $\mathbb{R}^3$. We can define the surface Σ_g as the oriented connected sum of g tori: $\Sigma_g := T^2\# \cdots \# T^2$ and the genus as the number of tori appearing in the sum.

It is very important that surfaces can be endowed with geometric structures, for example Riemannian metrics of constant curvature. (In dimension two "curvature" has only one possible meaning, since there is only one tangent 2-plane at every point, and its sectional curvature equals the Gaussian curvature K.) By the Gauss-Bonnet theorem we always have

$$\frac{1}{2\pi}\int_{\Sigma_g} \mathrm{K}\,\mathrm{dvol}_g = \chi(\Sigma_g) = 2 - 2g\ ,$$

so that the sign of the Euler characteristic equals the sign of the (average) curvature. By the classification of space form, a manifold of constant curvature is the quotient of a unique simply connected model space, on which the fundamental group acts by isometries. In the case of surfaces this leads to Table 1.

Instead of Riemannian metrics we can consider complex structures to impose a geometry on a surface. Every surface admits a complex structure, i.e. an atlas $\{(U_i, \varphi_i)\}$ such that the transition

genus	K	quotient
0	+1	S^2
1	0	$\mathbb{R}^2/\mathbb{Z}^2$
≥ 2	-1	$\mathbb{H}^2/\pi_1(\Sigma_g)$

Table 1: The geometries of surfaces

maps $\varphi_j \circ \varphi_i^{-1}$ are holomorphic whenever $U_j \bigcap U_i \neq \emptyset$. (A surface endowed with a complex structure is called a "Riemann surface".) By the uniformization theorem, the universal covering of a Riemann surface Σ_g is biholomorphic to either $\mathbb{C}P^1$, $\mathbb{C}$ or $\mathbb{H}^2$ (resp. when g equals 0, 1 or is greater than 1). This is essentially equivalent to the classification of space forms alluded to above.

3.1.3 Dimension 3

The 3-dimensional case is much more complicated than the previous two cases. The following is the first step towards a classification, imitating the decomposition of surfaces as connected sums of tori.

Theorem 3.1 (Kneser, Milnor)**.** *Every connected, smooth, closed, oriented 3-manifold has an essentially unique prime decomposition under connected sum, namely*

$$M = M_1 \# \cdots \# M_k$$

where each M_i is undecomposable.

A manifold is called "prime" or indecomposable if, whenever it is split as a connected sum, one of the summands must be a sphere. It turns out that for a prime 3-manifold the fundamental group is indecomposable as a free product, since any such decomposition of the fundamental group is induced by a connected sum decomposition of the manifold.

In dimension 2 there was only one prime manifold, the torus T^2. In dimension 3 there are many, starting with $S^1 \times S^2$, $S^1 \times T^2 = T^3$, or more generally $S^1 \times \Sigma_g$. Understanding surfaces through their geometric structures leads one hope that perhaps something similar can be done for 3-manifolds. Before that becomes possible, prime 3-manifolds have to broken up further into simpler pieces. This next step does not have an analogue in dimension 2.

Theorem 3.2 (Jaco-Shalen, Johannson)**.** *Every prime M contains finitely many distinct isotopy classes of π_1-injectively embedded tori $\varphi_i : T^2 \to M$ such that $M \setminus \{\coprod \varphi_i(T^2)\}$ has the property that any π_1-injectively embedded torus is boundary-parallel.*

Here an embedded torus $\varphi : T^2 \to M$ is π_1-injectively embedded if the induced map $\varphi_* :$

$\pi_1(T^2) \to \pi_1(M)$ is injective, and it is boundary-parallel if, given a collar neighborhood of the boundary of $M \setminus \{\coprod T_i^2\}$, it is isotopic to a torus in the collar.

When braking up a manifold under connected sum, one cuts along spheres, and then closes the resulting boundary components by discs. Now we cut along π_1-injectively embedded tori, and then consider the resulting non-closed pieces, either as compact manifolds with boundary, or as non-compact manifolds without boundary (by removing the boundary). We do not close the boundary (by solid tori, say), since there isn't a canonical way to do so. Finally, the pieces of this JSJ or torus decomposition can be endowed with geometric structures, some of which have constant curvature, but some of which do not:

Theorem 3.3 (Thurston Geometrization)**.** *Every piece of the JSJ decomposition of a prime manifold carries an essentially unique geometry modelled on one of the following spaces:*

$$S^3,\ \mathbb{R}^3,\ \mathbb{H}^3,\ S^2 \times \mathbb{R},\ \mathbb{H}^2 \times \mathbb{R},\ \mathrm{Sol}^3,\ \mathrm{Nil}^3,\ \widetilde{\mathrm{Sl}_2(\mathbb{R})}\ .$$

The first three model spaces have constant sectional curvature, the next two are products of lower-dimensional constant curvature spaces, and the last three are Lie groups endowed with left-invariant metrics. For a manifold to be modelled on one of these geometries means that the (possibly non-compact) manifold without boundary carries a complete Riemannian metric for which the universal covering is isometric to the model space. Therefore, the fundamental group of the manifold acts by isometries on the model space.

The geometrization of 3-manifolds was proposed by Thurston in the late 1970s. Thurston himself proved it for many 3-manifolds, and others built on his work for years, until the final remaining cases were disposed of by Perelman about fifteen years ago using Hamilton's Ricci flow.

Geometrization implies that the topology and geometry of 3-manifolds are controlled in a very precise way by the fundamental group. A special case is the famous

Corollary 3.4 (Poincaré Conjecture)**.** *If M is a simply connected, smooth, closed, oriented 3-manifold then M is diffeomorphic to S^3.*

Let us summarize the theory of manifolds in dimensions $n \leq 3$ by recording the following facts:

(1) Every topological manifold has a unique differentiable structure. (This is elementary in dimensions 1 and 2, but is difficult in dimension 3.)

(1') In particular, $\mathbb{R}^n$ has a unique differentiable structure.

(2) The topology of manifolds is controlled by the fundamental group.

(3) The topology of manifolds is controlled by geometry.

Before looking at 4-manifolds, let us look at the corresponding statements in dimensions $n \geq 5$. In these large dimensions we have:

(1) is false (first example: Milnor's exotic S^7)

(1') is true, essentially because smooth structures are classified by algebraic topology, and this is trivial on $\mathbb{R}^n$.

(2) is false, for example because there are many simply connected manifolds.

(3) is also false.

From now on we only consider 4-manifolds. In this dimension the corresponding statements are:

(1) is false, for instance $\mathbb{C}P^2 \# k\overline{\mathbb{C}P^2}$ admits exotic smooth structures for $k \geq 2$.

(1') is false, since $\mathbb{R}^4$ has exotic smooth structures, in fact, uncountably many!

(2) is false in several ways.

(3) is false, but nevertheless various geometric structures, like complex or symplectic ones, are intimately related to differential topology and are useful for understanding 4-manifolds.

3.2 Intersections forms of four-manifolds

Let us now begin the discussion of smooth, closed, connected, oriented 4-manifolds. The first examples that come to mind are:

- S^4, T^4;
- $\mathbb{R}^4/\Gamma$ and $\mathbb{H}^4/\Gamma$ where $\Gamma \subset \mathrm{Isom}(\mathbb{R}^4, g_0)$ respectively $\Gamma \subset \mathrm{Isom}(\mathbb{H}^4, g_{hyp})$;
- $N^3 \times S^1$ or more generally S^1-bundles over N^3 or N^3-bundles over S^1;
- $\Sigma_{g_1} \times \Sigma_{g_2}$ or more generally Σ_{g_1}-bundles over Σ_{g_2};
- $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$, with the latter being the same manifold as the first, but with the opposite orientation;
- connected sums of the manifolds above.

The simplest algebraic invariant of 4-manifolds is the **intersection form**. We can define it using the cup product in cohomology, followed by evaluation on the fundamental class:

$$\begin{aligned} H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) &\longrightarrow \mathbb{Z} \\ (\alpha, \beta) \qquad\qquad &\longmapsto < \alpha \cup \beta, [M] > \ . \end{aligned}$$

By linearity, the kernel must contain the torsion subgroup in $H^2(M;\mathbb{Z})$ and so we pass to the induced map

$$Q_M \colon H^2(M;\mathbb{Z})/_{\mathrm{Tor}} \times H^2(M;\mathbb{Z})/_{\mathrm{Tor}} \longrightarrow \mathbb{Z}\ ,$$

which is bilinear and symmetric as before, but is also nondegenerate by Poincaré duality.

To understand the presence of torsion in the (co)homology of M we look at the Universal Coefficient Theorem for cohomology. This gives us the exact sequence

$$1 \to \mathrm{Ext}(H_{i-1}(M;\mathbb{Z}),\mathbb{Z}) \to H^i(M;\mathbb{Z}) \to \mathrm{Hom}(H_i(M;\mathbb{Z}),\mathbb{Z}) \to 1 \; ,$$

which tells us that $H^1(M;\mathbb{Z})$ is torsion-free, and $\mathrm{Tor}(H^2(M;\mathbb{Z})) = \mathrm{Tor}(H_1(M;\mathbb{Z}))$. This last group may well be non-zero, but depends only on the fundamental group $\pi_1(M)$. Using Poincaré duality, we see that H_1, H^2, H_2 and H^3 all have the same torsion subgroups, whereas the other homology and cohomology groups of M are torsion-free.

Instead of considering the intersection form on cohomology, we can just as well consider it on homology. The Poincaré dual of the above definition is

$$\begin{aligned} Q_M \colon H_2(M;\mathbb{Z})/_{\mathrm{Tor}} \times H_2(M;\mathbb{Z})/_{\mathrm{Tor}} &\longrightarrow \mathbb{Z} \\ ([c],[d]) &\longmapsto c \cdot d \; , \end{aligned}$$

where $c \cdot d$ is the intersection number of the cycles c and d. Intersection numbers have a nice description when we consider submanifolds, rather than arbitrary cycles. In fact, every degree 2 homology class can be represented by an embedded surface. For the moment, we are content with the following weaker statement. A better statement will be proved in Proposition 3.34.

Lemma 3.5. *If M is a smooth simply connected* 4*-manifold, then every element of* $H_2(M;\mathbb{Z})$ *can be represented by an immersed sphere and by a smoothly embedded oriented surface of possibly higher genus.*

Proof. Since $\pi_1(M)$ is trivial the Hurewicz map $h : \pi_2(M) \to H_2(M;\mathbb{Z})$ is surjective. Thus for each class in $\alpha \in H_2(M;\mathbb{Z})$ we can find a map $f \colon S^2 \longrightarrow M$ such that $f_*[S^2] = \alpha$. We may assume that f is smooth, and, by transversality, an immersion. In order to get an embedding we modify the immersed sphere to remove the transverse double points. Locally, at these points, the sphere looks like the set of points $(z, w) \in \mathbb{C}^2$ such that $zw = 0$. So we replace $zw = 0$ by $zw = \varepsilon$ for a nonzero ε of small norm. □

For two homology classes represented by oriented embedded surfaces Σ_{g_1} and Σ_{g_2} we can calculate the intersection number geometrically. By transversality we may assume that the surfaces are transverse to each other. We give a sign ± 1 to each intersection point p_i, namely, since $T_{p_i}M = T_{p_i}\Sigma_{g_1} \oplus T_{p_i}\Sigma_{g_2}$, we assign $+1$ if the orientation of $T_{p_i}M$ is induced by those of $T_{p_i}\Sigma_{g_1}$ and $T_{p_i}\Sigma_{g_2}$ and -1 otherwise. Then the intersection number is

$$\Sigma_{g_1} \cdot \Sigma_{g_2} = \sum_{p_i} \pm 1 \; .$$

For the four-sphere S^4 the intersection form is trivial because $H_2(S^4) = 0$. The first nontrivial example is given by $\mathbb{C}P^2$. In fact $H_2(\mathbb{C}P^2, \mathbb{Z}) = \mathbb{Z}$ and it is generated by $[\mathbb{C}P^1]$. The intersection number $\mathbb{C}P^1 \cdot \mathbb{C}P^1 = +1$, because the orientation of $\mathbb{C}P^2$ is given by the complex structure, so that two projective lines intersect positively. Thus $Q_{\mathbb{C}P^2} = (+1)$, and if we reverse the orientation, then $Q_{\overline{\mathbb{C}P^2}} = (-1)$. In particular this implies that $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ are not oriented homotopy equivalent, in particular $\mathbb{C}P^2$ does not admit any orientation-reversing self-diffeomorphism.

Using the Mayer-Vietoris sequence, one can prove that the following relation holds:

$$Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2} \ .$$

Please carry out the proof as an exercise. (This is Exercise 3 on Sheet 4.)

Note that the intersection form on homology groups defines a homomorphism

$$\begin{array}{rccc} \varphi_M \colon & H_2(M;\mathbb{Z})/_{\mathrm{Tor}} & \longrightarrow & \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) \\ & [\Sigma] & \longmapsto & ([S] \mapsto S \cdot \Sigma) \end{array}$$

where the right hand group is $H^2(M;\mathbb{Z})/_{\mathrm{Tor}}$ by the Universal Coefficient Theorem. This homomorphism is an isomorphism by Poincaré duality, and so φ_M is invertible over $\mathbb{Z}$. This implies that Q_M is unimodular, i.e. $\det Q_M = \pm 1$.

We have already seen the following examples of intersection forms:

1. $Q_{S^4} = 0$, $Q_{\mathbb{C}P^2} = (1)$, $Q_{\overline{\mathbb{C}P^2}} = (-1)$;
2. $Q_{M_1 \# M_2} = Q_{M_1} \oplus Q_{M_2}$;
3. $Q_{\overline{M}} = -Q_M$.

Combining these, we see that for $M = p\mathbb{C}P^2 \# q\overline{\mathbb{C}P^2}$ we get

$$Q_M = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix} ,$$

with p positive and q negative entries on the diagonal, and zeroes everywhere else. For obvious reasons, we abbreviate this to $p(+1) \oplus q(-1)$. Thus intersection forms of all possible ranks and signatures (see Definition 3.6 below) can be realized as intersection forms of a connected sums of copies of $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$.

If we work over $\mathbb{R}$ instead of over $\mathbb{Z}$, then $Q_M \otimes_{\mathbb{Z}} \mathbb{R}$ is diagonalizable for all M, and so is equivalent to $Q_{p\mathbb{C}P^2 \# q\overline{\mathbb{C}P^2}}$ for suitable p and q. Thus p (resp. q) is the maximal dimension of a subspace of $H_2(M;\mathbb{R})$ on which the restriction of the intersection form is positive definite (resp. negative definite). We can now introduce the following

Definition 3.6. The *signature* $\sigma(M)$ of a 4-manifold M is defined by $\sigma(M) = p - q$.

From now on we will use the following notation:

$$b_2^+ = p;\;\; b_2^- = q;\;\; b_2 = \mathrm{rk}(H_2(M;\mathbb{Z})) = \dim_{\mathbb{R}}(H_2(M;\mathbb{R})) = b_2^+ + b_2^-;\;\; \sigma(M) = b_2^+ - b_2^- .$$

We already alluded to the following straightforward result from linear algebra.

Lemma 3.7. *Symmetric bilinear forms are classified over $\mathbb{R}$ by their rank and signature.*

We will see shortly that this does not hold over $\mathbb{Z}$.
Next we look at some important examples.

Example 3.8. Consider $M = S^2 \times S^2$, i.e. the trivial S^2-bundle over S^2. We know that

$$H_2(S^2 \times S^2, \mathbb{Z}) = \mathbb{Z}^2 ,$$

and we can choose $S^2 \times pt$ and $pt \times S^2$ as generators. It is clear that each of these generators has selfintersection 0 since we can move each factor off itself. What we know so far is then $Q_M = \left(\begin{smallmatrix} 0 & ? \\ ? & 0 \end{smallmatrix}\right)$. Moreover $? = \pm 1$ because the generators have only one intersection point, and it is transverse. Thus we can choose the orientation so that we get $Q_M = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Consider now the diffeomorphism

$$\begin{aligned} f : S^2 \times S^2 &\longrightarrow S^2 \times S^2 \\ (x, y) &\longmapsto (x, -y) \end{aligned}$$

where x and y are regarded as unit vectors in $\mathbb{R}^3$. This is clearly orientation reversing since it preserves the orientation of the first factor and reverses it in the second factor. Thus, unlike the case of $\mathbb{C}P^2$, there exists an orientation-reversing diffeomorphism of $S^2 \times S^2$. The map induced by f in homology

$$f_* : H_2(S^2 \times S^2, \mathbb{Z}) \to H_2(S^2 \times S^2, \mathbb{Z})$$

is an isomorphism between $Q_{S^2 \times S^2}$ and $Q_{\overline{S^2 \times S^2}} = -Q_{S^2 \times S^2}$.

From now on we will denote the intersection form $Q_{S^2 \times S^2}$ by H and refer to it as a hyperbolic pair.

Example 3.9. The second example we want to present is the non-trivial S^2-bundle over S^2, which we will denote by $S^2 \tilde{\times} S^2$. Consider any smooth S^2-bundle over S^2 with structure group $\mathrm{SO}(3)$. We can decompose the base S^2 into two hemispheres and, since they are both contractible, the bundle is trivial over each of them. Thus we may choose a trivialization on each hemisphere. Then M^4 is obtained by gluing together two copies of $D^2 \times S^2$ using a smooth transition map $g : \partial D^2 = S^1 \longrightarrow \mathrm{SO}(3)$, and M depends only on the homotopy class of g, i.e. $[g] \in \pi_1(\mathrm{SO}(3))$. Since this group is of order 2, we have exactly two S^2-bundles over S^2 with linear structure group, up to bundle isomorphism. However, we do not (yet) know, whether the two total spaces are perhaps diffeomorphic by a diffeomorphism that does not preserve the fibrations.

In order to study the (potentially) non-trivial bundle determined by the non-trivial element of $\pi_1(\mathrm{SO}(3))$ we regard it as the unit sphere vector bundle of a rank 3 vector bundle endowed with a metric and an orientation. Let $E \longrightarrow S^2$ be an oriented rank 2 bundle with Euler class $e(E) = a$ where a is a generator of $H^2(S^2; \mathbb{Z}) = \mathbb{Z}$. Let now $V := E \oplus \mathbb{R}$ and $M = S(V)$ be the sphere bundle of V. We now look at the intersection of M with the two direct summands of the vector bundle:

- $M \cap \mathbb{R}$ consists of two points in each fiber which make two disjoint copies of S^2 (since the $\mathbb{R}$ component of V is trivial). One can consider $M \cap \mathbb{R}$ as two sections of V;
- $M \cap E$ is a circle bundle over S^2.

Note that the two sections of $M \cap \mathbb{R}$ are separated by $M \cap E$, thus we obtain two disk bundles as tubular neighborhoods of the components of $M \cap \mathbb{R}$. Moreover the S^1-bundle over S^2 is the sphere bundle of E, which satisfies $e(E) = a$, and thus it is the Hopf fibration implying $M \cap E = S^3$. Up to choosing orientations, the two disk bundles have Euler number ± 1. We see that $M \setminus S^3$ has two components, each of them is a disk over S^2 with Euler number ± 1, thus each component is a copy of $\mathbb{C}P^2 \setminus B^4$ or $\overline{\mathbb{C}P^2 \setminus B^4}$.

We can now conclude that $S^2 \tilde{\times} S^2$ is one of the following manifolds: $\mathbb{C}P^2 \# \mathbb{C}P^2$, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ or $\overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$.

Now we will destinguish the two S^2-bundles using the following notion of parity.

Definition 3.10. A symmetric bilinear form Q_M over $\mathbb{Z}$ is called *even* if $Q_M(\alpha, \alpha) \equiv 0 \pmod 2$ for all α, and is called *odd* otherwise.

One can easily see that H is even by computing $H(a\alpha_1 + b\alpha_2, a\alpha_1 + b\alpha_2)$ where α_1 and α_2 are the generators of $H_2(S^2 \times S^2; \mathbb{Z})$ given by the two factors of selfintersection zero. Since the intersection form of $S^2 \tilde{\times} S^2$ is odd (it contains sections of selfintersection ± 1), we have proved that $S^2 \times S^2$ is not homotopy equivalent to $S^2 \tilde{\times} S^2$.

The parity of an intersection form plays an important role in the following classification result:

Theorem 3.11 (Hasse-Minkowski Classification). *If Q_M is indefinite, then it is equivalent either to $p(1) \oplus q(-1)$ with $p, q > 0$ if it is odd, or to $aH \oplus bE_8$ with $a \geq 1$ and $b \in \mathbb{Z}$ if it is even.*

Here E_8 is the incidence matrix of the Dynkin diagram of the exceptional Lie group E_8 depicted in Figure 1. Every circle in Figure 1 is a generator of $\mathbb{Z}^8$ of selfintersection -2 and two generators are connected by an edge if they have intersection number 1. The intersection form E_8 is then

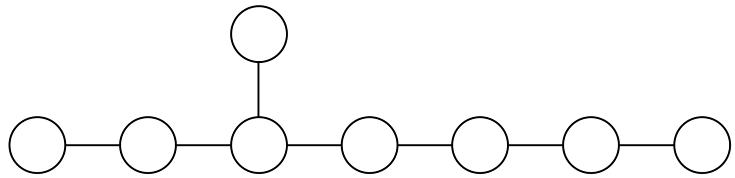


Figure 1: Dynkin diagram of E_8

given by:

$$E_8 = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{pmatrix}$$

Note that, since $\sigma(H) = 0$ and $\sigma(E_8) = -8$, the Hasse-Minkowski classification implies the following:

Corollary 3.12. *If Q_M is even then $\sigma(M) \equiv 0 \pmod 8$.*

Having already proved that the two S^2-bundles over S^2 are not diffeomorphic, in fact, not homotopy equivalent to each other, let us now identify the non-trivial bundle explicitly.

Lemma 3.13. *There is a diffeomorphism $S^2 \tilde{\times} S^2 \cong \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$.*

Proof. A fiber in $S^2 \tilde{\times} S^2$ is an embedded surface with selfintersection 0. Since it intersects a section of the bundle in one point, it represents a non-zero homology class, showing that the intersection form is indefinite. Thus $S^2 \tilde{\times} S^2$ must be diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ since the intersection forms of the other two candidates $\overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2 \# \mathbb{C}P^2$ are definite. □

3.2.1 Homotopy type and intersection form

We now give the homotopy classification of simply connected 4-manifolds. This shows the power of the intersection form.

Theorem 3.14 (Whitehead, Milnor)**.** *Two simply connected, smooth, closed, oriented* 4*-manifolds are oriented homotopy equivalent if and only if their intersection forms are isomorphic over* $\mathbb{Z}$*.*

Proof. If M_1 and M_2 are oriented-preservingly homotopy equivalent, then their intersection forms agree by homotopy invariance of (co)homoloy. The converse is more difficult. A first observation (proven using e.g. Morse theory) is that every compact smooth manifold has the homotopy type of a finite CW complex. This applies in particular to M_1 and M_2.

Let $M_0 = M \setminus e^4$ be the complement of a 4-cell in a simply connected closed four-manifold M. Then

$$H_k(M_0) = \begin{cases} \mathbb{Z} & k = 0 \\ 0 & k = 1, \geq 3 \\ \mathbb{Z}^m & k = 2 \end{cases}$$

where $m = b_2(M)$ since there is no torsion and the 3-skeleton of M and M_0 coincide. Because M is simply connected, $\pi_2(M)$ surjects onto $H_2(M_0)$ by Hurewicz' theorem. Pick a basis $\{e_j\}$ for $H_2(M_0)$ and continuous maps $f_j : S^2_j \to M_0$ such that $(f_j)_*[S^2_j] = e_j$. After homotoping some of the f_j's to ensure they all hit a fixed base point, they yield a map

$$f : \bigvee_{j=1}^{m} S^2 \to M_0$$

which induces an isomorphism on homology in every degree. It is a theorem due to Whitehead that such a map between simply connected CW complexes is a homotopy equivalence. We now have $M \simeq \bigvee_j S^2_j \cup_g e^4$, where $g : S^3 \to \bigvee_j S^2_j$ is the gluing map of the 4-cell. Therefore, the homotopy type of M is determined by $b_2(M)$ and the homotopy class of g. We start by considering some low values of m case-by-case:

(i) The case $m = 0$ is easy: g must be constant, hence $M \simeq S^4$.

(ii) If $m = 1$, we have a map $g : S^3 \to S^2$ hence $[g] \in \pi_3(S^2)$. So we need to figure out $\pi_3(S^2)$. We will do this via the Thom-Pontryagin construction. First, we will try to find a nice representative of $[g]$. Think of S^2, S^3 as embedded in $\mathbb{R}^3, \mathbb{R}^4$. For every continuous map $g : S^3 \to S^2$, we can find a smooth map $\tilde{g} : S^3 \to \mathbb{R}^3$ such that $\|\tilde{g}(x) - g(x)\| < \epsilon$ for arbitrarily small $\epsilon > 0$. Making ϵ small enough, we see that $\tilde{g}$ avoids $0 \in \mathbb{R}^3$. By "pushing radially", we see that $\tilde{g}$ is smoothly homotopic to a map into S^2; call the resulting map G.

Now G can be homotoped to g *outside* of the origin of $\mathbb{R}^3$: Because the points $G(x)$ and $g(x)$ are ϵ-close, the straight line connecting them never passes through the origin. Pushing this straight line into the sphere, we obtain a homotopy through maps into S^2 between g and G, showing that G is a smooth representative of $[g]$.

Now, we are ready to sketch the Pontryagin-Thom construction: Let $p \in S^2$ be a regular value of g, which we assume to be smooth from now on. Then $g^{-1}(p)$ is a smooth, compact one-dimensional submanifold of S^3, hence a union of circles. For each connected component, the normal bundle is a rank 2 oriented bundle which can be thought of as a tubular neighborhood of the circle. For every $q \in g^{-1}(p)$, $D_q g$ induces an isomorphism $T_q S^3 / T_q(g^{-1}(p)) \cong T_p S^2$. This defines a trivialization of the normal bundle $\nu(g^{-1}(p))$ in S^3 (whose fibers are exactly $T_q S^3 / T_q(g^{-1}(p)))$, since each fiber is identified with the same vector space $T_p S^2$. $g^{-1}(p)$ is called a *framed submanifold* (illustrated in figure 2).

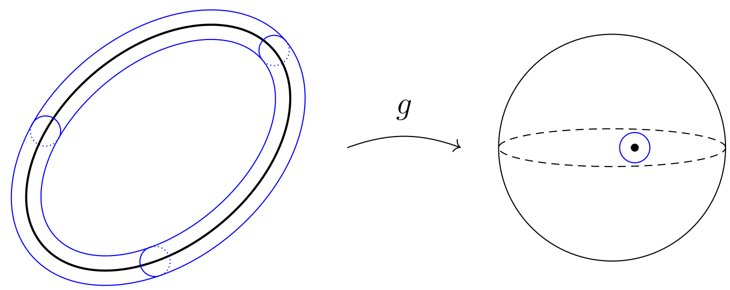


Figure 2: A framed circle in S^3

We would like to extract data that is independent of our initial choice of regular value. Pick a different regular value p' and connect the two by a path. The preimage of the path will, for a generic path, be an embedded surface (a bordism between $g^{-1}(p)$ and $g^{-1}(p')$) and the frame is also transferred, i.e. $g^{-1}(p)$ changes through a framed bordism (illustrated in figure 3). Thus, the framed bordism class is independent of our choice of regular value.

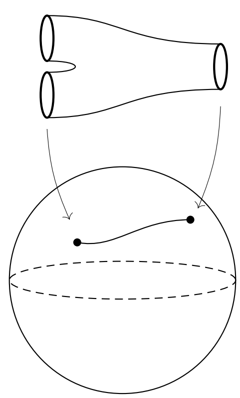


Figure 3: Picking two different regular values results in a framed cobordism between the preimages.

Now we consider homotopies of g. Let $H : S^3 \times I \to S^2$ be a homotopy from $g = H_0$ to $h = H_1$, which we can assume to be smooth (arguing as before). Pick a regular value p of g, h and H. Then $H^{-1}(p)$ is a surface in $S^3 \times I$ which projects under the canonical map $S^3 \times I \longrightarrow S^3$ to a framed cobordism between $g^{-1}(p)$ and $h^{-1}(p)$. Thus, the framed submanifold $g^{-1}(p)$ only depends on $[g]$ up to framed bordism.

The next step is to reverse the process, i.e. determine $[g]$ from a framed bordism class of framed submanifolds of S^3. Given a framed 1-dimensional submanifold $K \subset S^3$ (which is a link in general), we identify an open tubular neighborhood $T \supset K$ with $K \times D^2$, using the framing. Now define $g : S^3 \to S^2$ as follows: Let $x \in T \cong K \times D^2$ and set $g(x) = \mathrm{proj}_2(x) \in D^2 = S^2 \setminus \{-p\}$. For any $x \notin T$, set $g(x) = -p \in S^2$. Then the preimage of p is K, g is smooth near K (and can be homotoped to be smooth everywhere), p is a regular value and the induced framing is the right one. This establishes a bijection

$$\pi_3(S^2) \longleftrightarrow \text{framed bordism classes of 1-dimensional submanifolds of } S^3$$

and completes the Thom-Pontryagin construction. It is a fact, which we do not prove, that every equivalence class can be represented by a connected, unknotted $S^1 \subset S^3$. All bordism classes are therefore only distinguished by framings. Two different framings of $S^1 \subset S^3$ differ by a map $\rho : S^1 \to SO(2)$—identify the fibers of $\nu(S^1)$, i.e. disks, over one point and then track how the two framings differ by a rotation at each point. But of course $SO(2) \cong S^1$, hence $\pi_3(S^2) \cong \pi_1(S^1) = \mathbb{Z}$. This finally allows us to discuss the possible homotopy types of M for $b_2(M) = 1$:

a) $g \simeq$ const. Then $M \simeq S^2 \vee S^4$ but this cannot be the homotopy type of any closed, oriented manifold: It does not satisfy Poincaré duality since the intersection form is degenerate.
b) If g corresponds to a generator of $\pi_3(S^2)$, the attaching map is the Hopf fibration. But this yields the standard description of the CW structure of $\mathbb{C}\mathrm{P}^2$, i.e. $M \simeq \mathbb{C}\mathrm{P}^2$ or $\overline{\mathbb{C}\mathrm{P}^2}$.
c) If $[g] = \lambda \in \mathbb{Z}$ with $|\lambda| \geq 2$ (λ is called the linking number), then $Q_M = (\pm\lambda)$, but this is not unimodular. Thus, this can never be the homotopy type of a closed, oriented manifold.

We conclude that for $m = 1$, there are precisely two possibilities, distinguished by their orientations. Now, we turn to $b_2(M) = m \geq 2$. Then up to homotopy, we may take $g : S^3 \to \bigvee_j S_j^2$ to be smooth on the preimage of one hemisphere (not containing the "shared point" of the spheres) of each copy of S^2. Pick a regular value $p_j \in S_j^2$ and set $K_j := g^{-1}(p_j)$. This is again a framed 1-dimensional submanifold of S^3 and as before, $[g]$ corresponds to the framed bordism class of $K_1 \sqcup \cdots \sqcup K_m$.

This corresponds to an $(m \times m)$-matrix of linking numbers of the circles representing the K_j's (recall that up to framed bordism, we may take the K_j's to be unknotted circles). We will argue that this is the intersection form. every circle K_j bounds a surface $\Sigma_j \subset B^4$ (push a Seifert surface in S^3 into the 4-ball) and each Σ_j defines a closed surface by collapsing the boundary circle. In particular, when we glue onto $\bigvee_j S_j^2$, the boundary is collapsed, i.e. each Σ_j yields a closed surface $\tilde{\Sigma}_j$. These $\tilde{\Sigma}_j$'s yield a basis for $H_2(\bigvee_j S_j^2 \cup_g B^4)$: Since they intersect (only) the j-th copy of S^2 exactly in one point this is a "dual basis" to $\{S_j^2\}$. If one computes $\tilde{\Sigma}_i \cdot \tilde{\Sigma}_j$, it turns out to be the linking number $\mathrm{lk}(K_i, K_j)$. This shows that $(Q_M)_{ij} = \mathrm{lk}(K_i, K_j)$.

Requiring unimodularity in order to obtain the homotopy type of a manifold, we find all possibilities. Conversely, the intersection form tells us the linking numbers, which determine how the manifold is constructed and, in particular, the homotopy class $[g]$.

$\square$

3.2.2 Further classification results and a conjecture

We have seen that the homotopy type of a simply connected four-manifold is determined by its intersection form. However, we do not know which unimodular forms actually arise from closed smooth manifolds. First, we have the following result, which in particular says that the intersection form determines not just the homotopy type, but also the homeomorphism type. Moreover, all possible forms arise for topological (not necessarily smooth) manifolds.

Theorem 3.15 (Freedman 1982)**.** *1. Two smooth closed oriented simply connected 4-manifolds are homeomorphic if and only if their intersection forms are isomorphic over $\mathbb{Z}$.*

2. For every unimodular symmetric bilinear form over $\mathbb{Z}$ there exists a simply connected closed oriented topological 4-manifold whose intersection form is the given one.

The second part is in marked contrast with the following result, which was the first application of gauge theory to four-manifold topology:

Theorem 3.16 (Donaldson 1983)**.** *If the intersection form of a smooth closed oriented 4-manifold is definite, then it is equivalent over $\mathbb{Z}$ to a diagonal form, i.e. $\pm \oplus p(1)$.*

Recall that we discussed Donaldson's proof using the 1-instanton moduli space from Yang-Mills theory at the end of the course last semester. We will give a proof using the Seiberg-Witten equations later in this course.

The combination of the results of Freedman and Donaldson show that there exist many definite intersection forms which do not arise from smooth manifolds although they are realized by simply connected topological manifolds. This gives many examples of topological manifolds without any smooth structure. We also see that every definite intersection form that occurs for a smooth manifold also occurs for $\#p\mathbb{C}P^2$ or $\#p\overline{\mathbb{C}P^2}$ for a suitable p.

Recall that any indefinite even form is equivalent, by the Hasse-Minkowski classification, either to a diagonal form (if it is odd), or to $aH \oplus bE_8$ (if it is even). The odd diagonal forms are all realized by connected sums of the form $\#p\mathbb{C}P^2\#q\overline{\mathbb{C}P^2}$. Among the even forms we can certainly realize those with $b = 0$ by connected sums of copies of $S^2 \times S^2$. However, not all even forms can be realized by smooth manifolds. In this direction one has the following so-called $11/8$-Conjecture:

Conjecture 3.17. *If M is a closed oriented smooth simply connected 4-manifold and Q_M is even then*

$$b_2(M) \geq \frac{11}{8}|\sigma(M)| \ .$$

Writing the intersection form as $aH \oplus bE_8$, this inequality becomes $2a \geq 3|b|$.

Although the conjecture is still open in general, it is known to hold in some special cases. The best currently known inequality is $b_2(M) \geq \frac{10}{8}|\sigma(M)|$.

Remark 3.18. There is a compact connected oriented smooth 4-manifold M with $\pi_1(M) = \mathbb{Z}_2$

and $Q_M = H \oplus E_8$, showing that the assumption that M is simply connected is necessary in the Conjecture.

3.3 Non-trivial fundamental groups

Having so far discussed mostly simply connected manifolds, let us now consider questions about the fundamental group. Let M be a smooth closed oriented connected 4-manifold with possibly nontrivial fundamental group. From Morse theory we know that M has the homotopy type of a CW-complex. Since M is connected we can assume that the 0-skeleton $M^{(0)}$ consists in only one point. Hence the 1-skeleton is just the one-point-union of k copies of S^1: $M^{(1)} = S^1 \vee \cdots \vee S^1$. The fundamental group of the one-skeleton is the free group on k generators $F_k = \mathbb{Z} * \cdots * \mathbb{Z}$. The 2-skeleton is obtained from $M^{(1)}$ by attaching l 2-cells and it is then determined by the attaching maps $g_i : S^1 \longrightarrow M^{(1)}$. Thus the homotopy type of $M^{(2)}$ depends on the classes $[g_i] \in \pi_1(M^{(1)}) = F_k$. Now an element represented by g_i is trivial in $\pi_1(M^{(2)})$ hence we get

$$\pi_1(M^{(2)}) = F_k / << g_1, \ldots, g_l >> ,$$

where the double brackets denote the normal subgroup generated by the g_j. Finally since attaching cells of dimension ≥ 3 does not change the fundamental group we have $\pi_1(M^{(2)}) = \pi_1(M)$. This construction gives us a presentation of $\pi_1(M)$ where the generators are the 1-cells and the relations are provided by the attaching maps of the 2-cells. Note that since M is compact we get a finite presentation of its fundamental group.

Example 3.19. Let us consider the case of the torus: $M = T^2$. The torus can be realized by $T^2 = (S^1 \vee S^1) \cup_g B^2$. Here we have $[g] \in \pi_1(S^1 \vee S^1) = F_2 =< a, b >$. In particular the class represented by g is the class $aba^{-1}b^{-1}$ which implies $\pi_1(T^2) = F_2 / << aba^{-1}b^{-1} >>= \mathbb{Z}^2$.

Unlike in dimensions ≤ 3, there are no restrictions on fundamental groups of four-manifolds beyond finite presentability.

Theorem 3.20 (Dehn 1912)**.** *Every finitely presentable group Γ can be realized as the fundamental group of a smooth closed oriented connected 4-manifold.*

Proof. The outline of the proof is very simple. Choose a finite presentation of Γ, namely $\Gamma =< g_1, \ldots, g_k | r_1, \ldots, r_l >$. To this presentation corresponds a 2-dimensional finite CW-complex K such that $K^{(1)} = S^1 \vee \cdots \vee S^1$ and the 2-cells are attached according to the relations $r_1, \ldots, r_l$. So now we have $\pi_1(K) = \Gamma$. The presentation complex K can be embedded in $\mathbb{R}^5$. The boundary of a regular neighbourhood is the required 4-manifold.

Let us go into the details a little more. Given a finite presentation for a group Γ, we associated to it a CW-complex K. Instead we could consider a 2-dimensional simplicial complex K with fundamental group Γ. Now, in order to construct a four-manifold we give an explicit embedding f of K in $\mathbb{R}^5$.

Consider the curve

$$\begin{aligned} c:\mathbb{R}^+ &\longrightarrow \mathbb{R}^5 \\ t &\longmapsto (t,t^2,t^3,t^4,t^5) \ . \end{aligned}$$

Then we embed the 0-simplices $p_0,\dots,p_l$ defining $f(p_i)=c(i)$. If two 0-simplices p_i and p_j are connected by a 1-simplex we map this 1-simplex to the line segment $[c(i),c(j)]$. In the same way we map the 2-simplices using convex hulls of sets of vertices in $\mathbb{R}^5$. The curve we used to define this map has some remarkable properties, in particular the convex hull of three distinct points on the curve is disjoint from the convex hull of any other three distinct points on the curve. This property makes the map we defined an embedding. Now consider a small neighborhood $U_\epsilon(K)$ of the image of the embedding, which we still denote by K:

$$U_\epsilon(K)=\{x\in\mathbb{R}^5|\ d(x,K)\le\epsilon\} \ .$$

The 4-manifold we are searching is the boundary $\partial U_\epsilon(K)=M$ of the neighborhood $U_\epsilon(K)$:

$$M=\{x\in\mathbb{R}^5|\ d(x,K)=\epsilon\} \ .$$

This M is clearly closed and connected because so is K. Moreover M is a smooth separating hypersurface in $\mathbb{R}^5$, hence it is orientable.

Now we only have to show that $\pi_1(M)=\Gamma$. Since $U_\epsilon(K)$ retracts to K we have $\pi_1(U_\epsilon(K))=\pi_1(K)=\Gamma$. Consider the inclusion $i\colon M\longrightarrow U_\epsilon(K)$. We will show that the induced map $i_*\colon\pi_1(M)\longrightarrow\pi_1(U_\epsilon(K))$ is an isomorphism. Every class $[\gamma]\in\pi_1(U_\epsilon(K))$ is represented by a loop γ which we may suppose by transversality to be disjoint from K. Every such a loop is homotopic to one in the boundary and this proves the surjectivity of i_*. In order to prove the injectivity suppose $i_*[\gamma]=0$. This means γ is nullhomotopic in $U_\epsilon(K)$. Thus there exist a map $f\colon D^2\longrightarrow U_\epsilon(K)$ such that its restriction to the boundary $f:\partial D^2=S^1\longrightarrow U_\epsilon(K)$ is a parametrization of γ. We can assume that $f(D^2)$ does not intersect K and therefore f is homotopic to $\overline{f}:D^2\longrightarrow M$. This implies that γ is nullhomotopic in M.

This ends our discussion of the proof of Theorem 3.20. □

Corollary 3.21. *Closed connected oriented smooth manifolds of dimension* 4 *cannot be classified algorithmically.*

This is because finitely presentable groups cannot be classified algorithmically (Markov).

We now want to investigate the influence of the fundamental group on other topological invariants.

Definition 3.22. A manifold M is called *aspherical* if $\pi_k(M)=0$ for any $k\ge 2$.

In dimension 3 one knows that every prime closed orientable 3-manifold M other than $S^1\times S^2$ is aspherical. This follows from classical results in 3-dimensional topology.

Definition 3.23. Let Γ be any group. A connected cell complex $B\Gamma$ is called a *classifying space* for Γ if $\pi_1(B\Gamma) = \Gamma$ and $\pi_k(B\Gamma) = 0$ for $k \geq 2$.

Classifying spaces exist, and are unique up to homotopy equivalence. We will not discuss uniqueness, but will give a particular construction of a classifying space for every finitely presentable group by starting from a manifold with that group as fundamental group. Before doing that, let us mention that there exists a contractible cell complex $E\Gamma$ on which Γ acts freely and such that $B\Gamma = E\Gamma/\Gamma$. We can think of $E\Gamma$ as the universal covering of $B\Gamma$ or as a fibration over $B\Gamma$. Note that, by the long exact sequence of a fibration, saying that $E\Gamma$ is contractible is equivalent to saying that $B\Gamma$ is aspherical.

We will now provide a construction of the classifying space of a group Γ. Consider any manifold M with $\pi_1(M) = \Gamma$. If the higher homotopy groups vanish, then M is aspherical and therefore a $B\Gamma$. If M is not aspherical, then we have $\pi_k(M) \neq 0$ for some $k \geq 2$. Consider the smallest such k. Then there exist a non null-homotopic map $f\colon S^k \longrightarrow M$. We attach B^{k+1} along this map so that we obtain a cell complex in which the class $[f] \in \pi_k$ is null-homotopic. Iterating this operation for classes running over a basis for $\pi_k(M)$ we get a complex $M_{k+1} := M \cup B^{k+1} \cup \cdots$.

This operation did not change $\pi_1(M)$, but it trivializes $\pi_k(M_{k+1})$. Continue inductively attaching cells in higher dimensions in order to obtain a cell complex $B\Gamma$ such that $\pi_1(B\Gamma) = \Gamma$ and $\pi_k(B\Gamma) = 0$ for all $k \geq 2$. Note that in general this construction leads to an infinite dimensional cell complex.

The inclusion map $c\colon M \longrightarrow B\Gamma$, which we will call the classifying map of M, induces an isomorphism on π_1. Therefore the induced map $c_*\colon H_1(M;\mathbb{Z}) \longrightarrow H_1(B\Gamma;\mathbb{Z})$ is also an isomorphism and, dually, so is $c^*\colon H^1(B\Gamma;\mathbb{Z}) \longrightarrow H^1(M;\mathbb{Z})$.

Proposition 3.24. *The following sequence is exact:*

$$\pi_2(M) \xrightarrow{h} H_2(M;\mathbb{Z}) \xrightarrow{c_*} H_2(B\Gamma;\mathbb{Z}) \longrightarrow 0\,,$$

where h is the Hurewicz map.

This exact sequence is often called the **Hopf sequence**.

Sketch of the proof. In the construction of $B\Gamma$ no additional 2-homology is created since we attached balls of dimension at least three. Thus the map c_* is surjective.

We show now that $c_* \circ h = 0$. Consider any class $[g] \in \pi_2(M)$. Then we have $c_*(h[g]) = c_*(g_*[S^2]) = (c \circ g)_*[S^2]$. But since $B\Gamma$ is aspherical $c \circ g : S^2 \longrightarrow B\Gamma$ is null-homotopic and $(c \circ g)_*[S^2] = 0$ as claimed.

Finally we have to show that $\mathrm{Ker}(c_*) \subset \mathrm{im}(h)$. If $[\alpha] \in H_2(M;\mathbb{Z})$ and $c_*[\alpha] = 0$ then $c(\alpha) = \alpha \circ c$ is a singular 2-cycle in $B\Gamma$ and is the boundary of a 3-chain β in $B\Gamma$: $\beta \in C_3(B\Gamma)$. Thus β does not involve cell of dimension higher than 3 and this implies that $\alpha \in \mathrm{im}(h)$. $\square$

Corollary 3.25. $c^* : H^2(B\Gamma;\mathbb{Q}) \longrightarrow H^2(M;\mathbb{Q})$ *is injective.*

We now look at some explicit examples of classifying spaces.

Example 3.26. $M = \Sigma_g$ with $g > 0$. In this case the universal covering is $\mathbb{R}^2$ (when $g = 1$) or $\mathbb{H}^2$ (when $g > 1$). Hence Σ_g is aspherical because its universal covering is contractible.

Example 3.27. $M = \Sigma_0 = S^2$. We know that $\pi_2(S^2) = \mathbb{Z}$ is generated by $\mathrm{Id} : S^2 \longrightarrow S^2$. Thus we construct the classifying space by attaching a 3-ball B^3 to S^2 via the identity map on the boundary. The result of this operation is clearly B^3. Thus the classifying space for the trivial group has the homotopy type of a point.

Example 3.28. Every flat manifold has $\mathbb{R}^n$ as universal covering and thus is aspherical. For instance T^n is aspherical and hence the classifying space for $\mathbb{Z}^n$.

Example 3.29. The argument in the previous example applies also to hyperbolic manifolds. Therefore any hyperbolic manifold is the classifying space for its fundamental group.

Example 3.30. We now give an example of how classifying maps influence the intersection forms of four-manifolds. So assume M is a closed oriented 4-manifold with $\pi_1(M) = \mathbb{Z}^2$. Then the classifying map $c\colon M \longrightarrow B\mathbb{Z}^2 = T^2$ induces an injection $c^*\colon H^2(T^2;\mathbb{R}) = \mathbb{R} \longrightarrow H^2(M;\mathbb{R})$. Then the generator $[\omega]$ of $H^2(T^2;\mathbb{R})$ is sent to a generator $[c^*\omega]$ of $H^2(M;\mathbb{R})$.

Now we have $[c^*\omega] \smile [c^*\omega] = c^*([\omega] \smile [\omega]) = 0 \in H^4(M;\mathbb{R})$ since $0 = [\omega] \smile [\omega] \in H^4(T^2;\mathbb{R})$. This shows that $\mathrm{im}(c^*)$ is a one-dimensional isotropic subspace for the intersection form Q_M. Therefore Q_M is indefinite and $\mathrm{b}_2(M) \geq 2$.

This shows that in the context of Donaldson's theorem about definite intersection forms, one never has to consider manifolds with fundamental group $\mathbb{Z}^2$.

3.4 Characteristic classes and computations of examples

3.4.1 Euler class and the second Stiefel-Whitney class

Consider $E \to M$, an oriented rank 2 real vector bundle. Choose a metric h. We can choose fiberwise isometric, orientation-preserving local trivializations $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^2$. On $U_i \cap U_j$, we have the transition functions

$$\begin{aligned} \psi_j \circ \psi_i^{-1} : (U_i \cap U_j) \times \mathbb{R}^2 &\to (U_i \cap U_j) \times \mathbb{R}^2 \\ (x, v) &\mapsto (x, g_{ji}(x)v) \end{aligned}$$

where $g_{ji} : U_{ij} \to \mathrm{SO}(2) = S^1$ is smooth. The g_{ij} satisfy the cocycle conditions:

- $g_{ij} = g_{ji}^{-1}$
- $g_{ij} g_{jk} = g_{ik}$ on U_{ijk}

and therefore define a cohomology class $[g_{**}]$ in $\check{H}^1(M; S_{S^1})$. This is independent of the choice of metric h. As done before, we use the short exact sequence $0 \to \mathbb{Z} \to \mathbb{R} \to S^1 \to 1$ to induce a long exact sequence on the level of sheaf cohomology. Since $S_{\mathbb{R}}$ is a fine sheaf, we have an isomorphism $\delta : H^1(M; S_{S^1}) \cong H^2(M; S_{\mathbb{Z}})$.

Definition 3.31 (Euler class)**.** We call $\delta[g_{**}] \coloneqq e(E)$ the Euler class of E.

Remark 3.32. The ring homomorphism $\mathbb{Z} \hookrightarrow \mathbb{R}$ defines a map

$$\begin{aligned} H^2(M;\mathbb{Z}) &\longrightarrow H^2(M;\mathbb{R}) = H^2_{\text{dR}}(M) \\ e(E) &\longmapsto e(E)_{\mathbb{R}} \,. \end{aligned}$$

The latter class can also be defined in terms of curvature, as we did last semester.

By definition, it is clear that the Euler class classifies such bundles:

Proposition 3.33. *Two oriented rank 2 bundles $E,\ F \to M$ are orientation-preserving isomorphic if and only if $e(E) = e(F) \in H^2(M;\mathbb{Z})$.*

We list some fundamental properties of the Euler class:

- $e(E) = 0$ if and only if E is trivial.
- $e(\bar{E}) = -e(E)$.
- If $f : N \to M$ is an orientation-preserving smooth map and $E \to M$ a rank 2 oriented bundle, then $e(f^*E) = f^*e(E)$.

The following refines Lemma 3.5.

Proposition 3.34. *If M is a compact connected oriented smooth 4-manifold, then every $\alpha \in H_2(M;\mathbb{Z})$ is represented by a smoothly embedded surface.*

Proof. Let e be the Poincaré-dual of $\alpha \in H_2(M;\mathbb{Z})$ and $E \to M$ a smooth, oriented rank 2 vector bundle with $e(E) = e$. Let $s : M \to E$ be a smooth section that is transverse to the zero section $s_0(M) = M$. Thus, for every $p \in s(M) \cap s_0(M)$, $T_pM + T_ps(M) = T_pE$. Then the preimage $s^{-1}(0) = M \cap s(M)$ is a 2-dimensional smooth submanifold of M which inherits a natural orientation. It is a general fact about the Euler class that, given this setup, $\iota_*([S]) = \alpha \in H_2(M;\mathbb{Z})$. where $\iota : S \hookrightarrow M$ is the inclusion. Modulo torsion, this may be proven by showing that for any $\beta \in H_2(M;\mathbb{Z})$, $\alpha \cdot \beta = [S] \cdot \beta$ (but it holds true generally). ☐

Recall now that an oriented, real, rank 2 bundle E is the same thing as a complex line bundle L via the correspondence $SO(2) \cong U(1)$, i.e. $E \leftrightarrow L$, such that $L_{\mathbb{R}} = E$.

Definition 3.35 (First Chern class)**.** We define $c_1(L) \coloneqq e(L_{\mathbb{R}})$ to be the first Chern class of a complex line bundle L, where $L_{\mathbb{R}}$ is oriented by the complex structure.

We can reformulate definition 2.40:

Definition 3.36 (Second Stiefel-Whitney Class)**.** Let $E, F \to M$ be an oriented, real vector bundles. Then there exists a unique $w_2(E) \in H^2(M; \mathbb{Z}_2)$ such that

(i) If E is trivial, then $w_2(E) = 0$.

(ii) $w_2(E \oplus F) = w_2(E) + w_2(F)$.

(iii) If E has rank 2, then $w_2(E) = r(e(E))$, where $r : H^2(M; \mathbb{Z}) \to H^2(M; \mathbb{Z}_2)$ is *reduction modulo 2*.

(iv) If $f : N \to M$ is a smooth map then $w_2(f^*(E)) = f^*(w_2(E))$.

These properties define the second Stiefel-Whitney class.

Uniqueness comes from uniqueness of $c_1(E)$, and we will not discuss existence here. An additional property of w_2 is $w_2(E) = w_2(\bar{E})$.

Proposition 3.37. *If M is a compact connected oriented smooth 4-manifold then $w_2(TM) = 0$ implies Q_M is even. The converse implication holds if $H_1(M; \mathbb{Z})$ is free of 2-torsion.*

Proof. Let $\iota : \Sigma \hookrightarrow M$ be a smoothly embedded, oriented surface representing a given class in $H_2(M; \mathbb{Z})$. Then $\iota^*TM = TM|_\Sigma = T\Sigma \oplus \nu(\Sigma)$ and both summands are oriented rank two bundles. Using the defining properties of the second Stiefel-Whitney class, we have:

$$\iota^* w_2(TM) = w_2(\iota^* TM) = w_2(\iota^* TM) = w_2(T\Sigma) + w_2(\nu(\Sigma)) \ .$$

Evaluating on $[\Sigma]$, we find

$$\langle \iota^* w_2(TM), [\Sigma] \rangle = r\langle e(T\Sigma), [\Sigma] \rangle + r\langle e(\nu(\Sigma)), [\Sigma] \rangle = r(\chi(\Sigma)) + r(\Sigma \cdot \Sigma) = r(\Sigma \cdot \Sigma)$$

where we used that the Euler class of the tangent bundle evaluates on the fundamental class to the Euler characteristic $\chi(\Sigma) = 2 - 2g \equiv 0 \mod 2$ while the normal bundle of Σ can be viewed as a tubular neighborhood, hence the zero locus of a generic section is exactly the self-intersection of Σ.

The equation $\langle \iota^* w_2(TM), [\Sigma] \rangle = r(\Sigma \cdot \Sigma)$ makes it clear that if $w_2(TM) = 0$, the intersection form must be even, since every class is represented by an embedded surface. Conversely, if Q_M is even we see that $\langle \iota^* w_2(TM), [\Sigma] \rangle = 0$ for every embedded surface Σ. Using the universal coefficient theorem, the Ext-term vanishes if there is no 2-torsion, hence in this case we conclude that $w_2(TM) = 0$. □

Corollary 3.38. *For every class $\alpha \in H_2(M; \mathbb{Z})$ we have $\langle w_2(TM), \alpha \rangle \equiv \alpha \cdot \alpha \mod 2$.*

In particular, if M is simply connected, then $H_*(M; \mathbb{Z})$ is torsion-free so by the results of section 2.3.2 we have:

Corollary 3.39. *Let M be a closed oriented smooth connected and simply connected 4-manifold. Then M is* Spin *if and only if Q_M is even.*

The following important result about the existence of Spin^c structures is what gets Seiberg-Witten theory off the ground.

Theorem 3.40 (Whitney)**.** *For any compact connected oriented smooth 4-manifold M, there exists some $c \in H^2(M;\mathbb{Z})$ such that $r(c) = w_2(TM)$, where r is reduction modulo 2. Hence such manifolds always admit a* Spin^c *structure.*

Proof. By naturality of the universal coefficient theorem under reduction, we have a commutative ladder

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}) & \xrightarrow{f} & H^2(M;\mathbb{Z}) & \xrightarrow{g} & \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} & & \\ 0 & \longrightarrow & \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}_2) & \xrightarrow[f']{} & H^2(M;\mathbb{Z}_2) & \xrightarrow[g']{} & \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}_2) & \longrightarrow & 0 \end{array}$$

An element $\varphi \in \mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}_2)$ lifts under r if and only if $\varphi(t) = 0$ for every torsion element $t \in H_2(M;\mathbb{Z})$. The bilinearity of the intersection form guarantees that it kills all torsion. Indeed, if α were k-torsion but $\alpha \cdot \alpha \neq 0$, we would have $(k\alpha)\cdot(k\alpha) = 0 = k^2(\alpha\cdot\alpha) \neq 0$. Since $Q_M(\alpha,\alpha) \equiv \langle w_2(TM),\alpha\rangle \mod 2$, we see that they agree as elements of $\mathrm{Hom}(H_2(M;\mathbb{Z}),\mathbb{Z}_2)$, i.e. $g'(w_2(TM)) = \langle w_2(TM), -\rangle =: \omega$ must lift.

By surjectivity of the top-right horizontal arrow, there exists some $x \in H^2(M;\mathbb{Z})$ such that $r(g(x)) = \omega$. Commutativity of the square tells us that $g'(r(x)) = \omega = g'(w_2(TM))$. But then exactness of the bottom row tells us that there exists some $\gamma \in \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z}_2)$ such that $f'(\gamma) = r(x) - w_2(TM)$. The first vertical map is surjective by general homological algebra arguments, hence there is some $\kappa \in \mathrm{Ext}(H_1(M;\mathbb{Z}),\mathbb{Z})$ such that $r(\kappa) = \gamma$, hence $f'(\gamma) = f(r(\kappa)) = r(f(\kappa))$. Now set $c = x - f(\kappa)$. Then $r(c) = r(x) - r(f(\kappa)) = r(x) - f(\gamma) = w_2(TM)$, hence c is an integral lift of $w_2(TM)$. □

3.4.2 Chern classes

Definition 3.41 (Chern classes)**.** If L is a complex line bundle, the *total Chern class* of L is $c(L) = 1 + c_1(L)$. If $E = L_1 \oplus L_2 \oplus \ldots \oplus L_k$ is a direct sum of complex line bundles, we extend the above definition in the obvious way:

$$c(E) \coloneqq (1 + c_1(L_1)) \smile (1 + c_1(L_2)) \smile \ldots \smile (1 + c_1(L_k))$$

Expanding the above, we obtain the *Chern classes* $c_i(E)$:

$$c(E) = 1 + \underbrace{\sum_{i=1}^{k} c_1(L_i)}_{c_1(E)\in H^2(M;\mathbb{Z})} + \underbrace{\sum_{1\le i<j\le k} c_1(L_i) \smile c_1(L_j)}_{c_2(E)\in H^4(M;\mathbb{Z})} + \ldots + \underbrace{\prod_{i=1}^{k} c_1(L_i)}_{c_k(E)\in H^{2k}(M;\mathbb{Z})}$$

The following proposition gives us a way to generalize the definition to arbitrary complex vector bundles:

Proposition 3.42. *For every complex vector bundle $E \to M$, there exists a so-called "splitting manifold" $f : N \to M$ with the following properties*

(i) $f^*E \cong L_1 \oplus \ldots \oplus L_k$*, where the L_i are line bundles.*

(ii) f^* *is injective on* $H^*(M;\mathbb{Z})$.

Sketch of proof. Set $n = \operatorname{rank}_{\mathbb{C}} E$ and consider the *projectivized bundle* $\pi : \mathbb{P}(E) \to M$, which is the $\mathbb{C}\mathrm{P}^{n-1}$-bundle over M with $(\mathbb{P}(E))_p = \mathbb{P}(E_p)$. The transition functions act on fibers by the action of $\mathrm{GL}(n,\mathbb{C})$ on $\mathbb{C}\mathrm{P}^{n-1}$ (the action descends from the action on $\mathbb{C}^n$). Now consider the pullback bundle π^*E, which yields the following diagram:

$$\begin{array}{ccccc} L_1 & \subset & \pi^*E & \longrightarrow & E \\ & \searrow & \downarrow & & \downarrow p \\ & & \mathbb{P}(E) & \xrightarrow[\pi]{} & M \end{array}$$

where L_1 is the tautological line bundle

$$L = \{(\ell, v) \in \mathbb{P}(E) \times E \mid v \in \ell\} \subset \{(\ell, v) \in \mathbb{P}(E) \times E \mid \pi(\ell) = p(v)\} = \pi^*E$$

Then $\pi^*E \cong L_1 \oplus Q$ where Q is a complement $Q \cong \pi^*E/L_1$. Iterating this process, we get a tower of projectivizations such that eventually $f^*E \cong \bigoplus_j L_j$. Injectivity on the level of cohomology follows from the Leray-Hirsch theorem. □

Definition 3.43. This allows us to define $c(E)$ for an arbitrary complex vector bundle as the unique element of $H^*(M;\mathbb{Z})$ that maps to $c(f^*E) = c(L_1 \oplus \cdots \oplus L_k)$ under f^*.

Remark 3.44. Of course, one should really check that using or deriving identities involving Chern classes does not take one out of the image of f^*. This can be done inductively by carefully using the (proof of the) Leray-Hirsch theorem.

This method of defining Chern classes in terms of split vector bundles and the techniques that it enables one to make use of collectively embody the so-called *splitting principle*. The basic properties of the Chern classes are:

(i) If $E \to M$ is trivial, then $c_i(E) = 0$ for all $i > 0$.

(ii) $c(E \oplus F) = c(E) \cdot c(F)$.

(iii) $w_2(E_{\mathbb{R}}) = r(c_1(E))$.

(iv) $c_i(f^*E) = f^*c_i(E)$ for all i.

(v) $c_i(\bar{E}) = (-1)^i c_i(E)$.

(vi) $c_i(E) = 0$ for $i > \operatorname{rank}_{\mathbb{C}} E$.

(vii) $c_i(E) = e(E_{\mathbb{R}})$ for $i = \operatorname{rank}_{\mathbb{C}} E$.

The last statement uses the Euler class for higher-rank bundles, which was not defined in this course. This is a characteristic class of oriented real bundles, so one considers the real bundle $E_{\mathbb{R}}$ underlying the complex E, and gives it the orientation defined by the complex structure.

Note that we omit the cup product in our notation. Since Chern classes commute, one can think of the cup product as multiplication of polynomials. We will use the standard notation $e(M) = e(TM)$, and $w_2(M) = w_2(TM)$ from now on. If M is an almost complex manifold, $c_i(M) = c_i(TM)$. However, note that the Chern classes of M depend on the choice of an almost complex structure (so one should properly write $c_i(M, J)$), but confusion rarely arises.

Lemma 3.45. *$H^*(\mathbb{C}\mathrm{P}^n;\mathbb{Z}) \cong \mathbb{Z}[x]/(x^{n+1} = 0)$, where $x \in H^2(M;\mathbb{Z})$ is of degree two, and $\mathbb{Z}[x]/(x^{n+1} = 0) = \{a_0 + a_1x^1 + a_2x^2 + \dots a_nx^n| \mid a_i \in \mathbb{Z}\}$ is the truncated polynomial ring. Moreover,*

$$c(\mathbb{C}\mathrm{P}^n) = (1+x)^{n+1} = \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{k=0}^{n} c_k(\mathbb{C}\mathrm{P}^n)\ .$$

Proof. The first statement is proven using cellular homology for the additive structure, while the multiplicative structure is determined as follows. The standard embedding $\mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{C}\mathrm{P}^2$ yields the positive generator of $H^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ and the Poincaré dual $[\mathbb{C}\mathrm{P}^1]$ self-intersects once. Thus, the cup product of the generator of $H^2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$ with itself yields the positive generator of $H^4(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$. Proceeding inductively along those lines yields the claim.

For the second part, let L be the tautological line bundle over $\mathbb{C}\mathrm{P}^n$. Then by linear algebra arguments, $T\mathbb{C}\mathrm{P}^n \oplus \underline{\mathbb{C}} \cong \bigoplus_{j=0}^n \bar{L}$. Hence $c(T\mathbb{C}\mathrm{P}^n) = (1+x)^{n+1}$. □

Example 3.46 ($M = \mathbb{C}\mathrm{P}^2$)**.** From the above formula, $c_1 = 3x$, $c_2 = 3x^2$. Since $T\mathbb{C}\mathrm{P}^2$ is of complex dimension 2, $c_2(\mathbb{C}\mathrm{P}^2) = e(\mathbb{C}\mathrm{P}^2)$, i.e.

$$\langle c_2(\mathbb{C}\mathrm{P}^2), [\mathbb{C}\mathrm{P}^2]\rangle = \langle e(\mathbb{C}\mathrm{P}^2), [\mathbb{C}\mathrm{P}^2]\rangle = \chi(\mathbb{C}\mathrm{P}^2) = 3 = b_0 + b_2 + b_4\ .$$

The final result in this section is concerned with surfaces embedded in almost complex 4-manifolds:

Theorem 3.47 (Adjunction formula)**.** *Let M be an oriented, smooth 4-manifold with an almost complex structure*[4] *J compatible with the orientation. Let $\iota : \Sigma \to M$ be a smoothly embedded surface with $J(T\Sigma) = T\Sigma$, i.e. Σ is an almost complex submanifold. Then the genus of Σ is given by*

$$g(\Sigma) = 1 + \frac{1}{2}\Big(\Sigma\cdot\Sigma - \langle c_1(M), \iota_*[\Sigma]\rangle\Big)\ .$$

[4] A $J \in \Gamma(\mathrm{End}\ TM)$ such that $J_p^2 = -\mathrm{Id}_{T_pM}$ for each $p \in M$ is said to be an almost complex structure for TM.

Proof. Since Σ is a J-holomorphic submanifold, we know that as complex vector bundles

$$TM|_\Sigma = T\Sigma \oplus \nu(\Sigma) \ .$$

Hence, we have

$$\iota^* c_1(M) = c_1\left(TM|_\Sigma\right) = c_1(T\Sigma) + c_1(\nu(\Sigma)) = e(T\Sigma) + e(\nu(\Sigma))$$

whence we can compute

$$\langle c_1(M), \iota_*[\Sigma]\rangle = \chi(\Sigma) + \Sigma\cdot\Sigma = 2 - 2g(\Sigma) + \Sigma\cdot\Sigma \ .$$

□

In particular, we see that the genus of a J-holomorphic curve is determined by its homology class. This applies in particular to smooth holomorphic curves in complex manifolds. It is of course false for smooth surfaces, since once can always add small inessential handles to increase the genus artificially without changing the homology class. The adjunction formula shows that this is not possible in the J-holomorphic setting. We will see later, that as a consequence of gauge theory, the genus of a smoothly embedded surface can not be reduced below certain lower bounds, which have a shape very much like the adjunction formula.

Example 3.48. A holomorphic curve of degree d, $\Sigma_d \subset \mathbb{C}\mathrm{P}^2$, is a smooth holomorphic curve of degree d, i.e. $[\Sigma_d] = d\cdot[\mathbb{C}\mathrm{P}^1] \in H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$, where $[\mathbb{C}\mathrm{P}^1]$ is the generator of $H_2(\mathbb{C}\mathrm{P}^2;\mathbb{Z})$. Therefore, $\Sigma\cdot\Sigma = d^2$ and by the adjunction formula,

$$g(\Sigma_d) = 1 + \frac{1}{2}(d^2 - 3\langle x, [\Sigma_d]\rangle) = \frac{1}{2}(d^2 - 3d + 2) = \frac{1}{2}(d-1)(d-2) \ .$$

This is known as the *degree formula*.

3.4.3 Pontryagin Classes

Definition 3.49 (Pontryagin Classes)**.** Let $V \to M$ be a real vector bundle. We define the Pontryagin classes of V by

$$p_i(V) := (-1)^i c_{2i}(V \otimes_{\mathbb{R}} \mathbb{C}) \in H^{4i}(M;\mathbb{Z}) \ .$$

The *total Pontryagin class* of V is $p(V) = \sum_j p_j(V)$.

The Pontryagin classes inherit all the properties of the Chern classes. Moreover, note that $\mathrm{rank}_{\mathbb{R}} V = \mathrm{rank}_{\mathbb{C}}(V\otimes_{\mathbb{R}}\mathbb{C})$, hence $p_i = 0$ if $2i > \mathrm{rank}_{\mathbb{R}} V$.

Example 3.50.

(i) $p_i(V) = 0$ for all i if $\mathrm{rank}_{\mathbb{R}} V = 1$.

(ii) Assume that $\mathrm{rank}_{\mathbb{R}} V = 2$. Then $p_i(V) = 0$ for $i \geq 2$, and $p_1(V) = -c_2(V \otimes_{\mathbb{R}} \mathbb{C})$. If V is orientable, we fix an orientation and henceforth think of V as a complex line bundle L. For a complex vector bundle E, $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong E \oplus \bar{E}$. Then

$$\begin{aligned} c_{2i}(E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}) &= c_{2i}(E \oplus \bar{E}) = c_{2i}(E) + c_{2i-1}(E)c_1(\bar{E}) + \ldots + c_1(E)c_{2i-1}(\bar{E}) + c_{2i}(\bar{E}) \\ &= c_{2i}(E) - c_{2i-1}(E)c_1(E) + c_{2i-2}(E)c_2(E) - \cdots - c_1(E)c_{2i-1}(E) + c_{2i}(E) \end{aligned}$$

Hence, in the case where $E = V$ is orientable and of real-rank 2, we fix an orientation and think of V as a complex line bundle L. $c_1(L)$ is then defined, and we have the important relation

$$p_1(V) = -c_2(L \oplus \bar{L}) = c_1^2(L) = e^2(V) \ .$$

Theorem 3.51 (Signature formula of Thom and Hirzebruch)**.** *For a compact connected smooth oriented* 4*-manifold* M*, the signature is given by*

$$\sigma(M) = \frac{1}{3}\langle p_1(TM), [M]\rangle \ .$$

Corollary 3.52.

(i) $\sigma(M) = 0$ *if and only if* $p_1(TM) = 0$.

(ii) $p_1(M)$ *is a multiple of 3 since* $\sigma(M) \in \mathbb{Z}$.

Example 3.53 (Surfaces in $\mathbb{C}\mathrm{P}^3$)**.** We discuss the analog of holomorphic curves in $\mathbb{C}\mathrm{P}^2$ in one dimension higher, namely algebraic surfaces in $\mathbb{C}\mathrm{P}^3$. Consider $\iota : X_d \hookrightarrow \mathbb{C}\mathrm{P}^3$, a smooth algebraic surface of degree d. This means that X_d is the zero locus of a generic homogeneous polynomial of degree d in four variables, the homogeneous coordinates of $\mathbb{C}\mathrm{P}^3$. Let $x \in H^2(\mathbb{C}\mathrm{P}^3, \mathbb{Z})$ be the positive generator $\langle x, [\mathbb{C}\mathrm{P}^1]\rangle = 1$. As before, we have $T\mathbb{C}\mathrm{P}^3|_{X_d} = TX_d \oplus \nu(X_d)$, since X_d is a holomorphic submanifold. Then

$$\begin{aligned} &\iota^* c(T\mathbb{C}\mathrm{P}^3) = c(TX_d) \smile c(\nu(X_d)) \\ \implies\ &\iota^*(1+x)^4 = (1 + c_1(TX_d) + c_2(TX_d))(1 + c_1(\nu(X_d))) \ . \end{aligned}$$

Equating the polynomials degree-by-degree, we see that

$$\begin{aligned} \iota^*(4x) &= c_1(TX_d) + c_1(\nu(X_d)) \\ \iota^*(6x^2) &= c_2(TX_d) + c_1(TX_d)c_1(\nu(X_d)) \ . \end{aligned}$$

Now, it is a fact from complex geometry that that $\nu(X_d) = \mathcal{L} = \mathcal{O}(d)|_{X_d}$, where $\mathcal{O}(d) \to \mathbb{C}\mathrm{P}^3$ is the holomorphic line bundle with $c_1(\mathcal{L}) = d \cdot x$. Hence,

$$\begin{aligned} &\iota^*(4x) = c_1(TX_d) + c_1(\iota^*\mathcal{L}) = c_1(TX_d) + \iota^*(d \cdot x) \\ \implies\ &c_1(TX_d) = \iota^*((4-d)x) = (4-d)\iota^* x \ . \end{aligned}$$

Using this in the equation for c_2, we get

$$c_2(TX_d) = (d(d-4)+6)\iota^*(x^2) \ .$$

We can now compute the Euler characteristic:

$$\begin{aligned}\chi(X_d) &= \langle c_2(TX_d), [X_d]\rangle = (d(d-4)+6)\langle \iota^* x^2, [X_d]\rangle \\ &= d(d^2-4d+6)\end{aligned}$$

where we used that $[X_d] = d[\mathbb{C}\mathrm{P}^2]$. The first Pontryagin class is now easy to compute:

$$\begin{aligned}p_1(TX_d) &= -c_2(TX_d \otimes_{\mathbb{R}} \mathbb{C}) = -c_2(TX_d \oplus \overline{TX}_d) \\ &= -2c_2(TX_d) + c_1^2(TX) = \iota^*((-2(d(d-4)+6)) + (4-d)^2)x^2) \\ &= (4-d^2)\iota^* x^2 \ .\end{aligned}$$

By the signature formula,

$$\sigma(X_d) = \frac{1}{3}(4-d^2)\langle \iota^* x^2, [X_d]\rangle = \frac{1}{3}d(4-d^2) \ .$$

Note that this is indeed an integer. If $d \equiv 0 \mod 3$ this is clear, while if $d \equiv \pm 1 \mod 3$ then $d^2 = 1 \mod 3$, hence $4 - d^2 \equiv 0 \mod 3$. To complete our analysis we use the *Lefschetz hyperplane theorem*, which implies that $\pi_1(X_d) = 1$. This implies that $b_1(X_d) = 0 = b_3(X_d)$, which in turn implies that $b_2(X_d) = \chi(X_d) - 2$. Since we also know $\sigma(M)$, we can now determine $b_2^{\pm}(M)$.

$$b_2^{\pm}(X_d) = \frac{1}{2}(b_2(X_d) \pm \sigma(X_d)) = \frac{1}{2}(\chi(X_d) - 2 \pm \sigma(M)) = \frac{d}{2}\Big(d^2 - 4d + 6 \pm \frac{1}{3}(4-d^2)\Big) - 1$$

Let us investigate the situation for low values of d:

(i) $d = 1$ yields $\mathbb{C}\mathrm{P}^2$; $\chi(X_1) = 3$ and $\sigma(X_1) = 1$.

(ii) For $d = 2$, $\chi(X_2) = 4$ and $\sigma(X_2) = 0$.

Lemma 3.54. *X_2 is diffeomorphic to $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 = S^2 \times S^2$.*

Proof. Consider the so-called *Segre map*

$$\begin{aligned} f : \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 &\longrightarrow \mathbb{C}\mathrm{P}^3 \\ ([x:y],[z:w]) &\longmapsto (xz : xw : yz : yw) \ . \end{aligned}$$

This is a well-defined holomorphic map, injective and in fact an immersion, hence an embedding. The image is precisely the zero-locus of the homogeneous second degree polynomial $f : \mathbb{C}^4 \to \mathbb{C}$ given by $f(t_0, t_1, t_2, t_3) = t_0 t_3 - t_1 t_2$. Any other (generic) degree 2 polynomial can be deformed to it, hence $X_2 \cong S^2 \times S^2$. □

(iii) For $d = 3$ we have $\chi(X_3) = 9$ and $\sigma(M) = -5$. The Hasse-Minkowski classification and Freedman's theorem 3.15 tell us that this manifold is homeomorphic to $\mathbb{C}\mathrm{P}^2\#6\overline{\mathbb{C}\mathrm{P}^2}$. In fact, the two manifolds are diffeomorphic, but we will not prove that.

(iv) $d = 4$ yields $\chi(X_4) = 24$ and $\sigma(X_4) = -16$. By the Hasse-Minkowski classification, the intersection form is determined by the parity of Q_M. Since there is no torsion in this case, this is determined by $w_2(X_d) = r(c_1(X_d)) = r((4-d)\iota^*x) \equiv d\,\iota^*x \mod 2$. Hence for $d = 4$, Q_{X_4} is even and $Q_{X_4} = 3H \oplus 2E_8$ by Hasse-Minkowski.

Definition 3.55 ($K3$ Surface)**.** A $K3$ surface X is a compact, complex surface with $\pi_1(X) = 0$ and $c_1(X) = 0$.

It is a fact, though not easy to prove, that any two K3 surfaces are diffeomorphic. Observe that for a $K3$ surface the $\frac{11}{8}$ inequality is sharp, that is, it is an equality in this case.

We summarize these results in a table:

	$d=1$	$d=2$	$d=3$	$d=4$
$\chi(X_d)$	3	4	9	24
$\sigma(X_d)$	1	0	-5	-16
Diffeomorphic to	$\mathbb{C}\mathrm{P}^2$	$S^2 \times S^2$	$\mathbb{C}\mathrm{P}^2\#6\overline{\mathbb{C}\mathrm{P}^2}$	$K3$

Regarding the $\frac{11}{8}$-conjecture, we can say a little bit more:

Proposition 3.56. *The $\frac{11}{8}$-conjecture is equivalent to the following statement: Every simply connected compact connected oriented smooth 4-manifold with even intersection form is homeomorphic to a connected sum of copies of $K3$, $\overline{K3}$ and $S^2 \times S^2$.*

Proof. Suppose that our 4-manifold M is homeomorphic to a connected sum of copies of $K3$, $\overline{K3}$ and $S^2 \times S^2$, i.e. $M = a\#b\overline{K3}\#c(S^2 \times S^2)$. Then

$$\begin{aligned} b_2(M) &= 22(a+b) + 2c \\ \sigma(M) &= 16(b-a)\;. \end{aligned}$$

So we compute

$$\frac{11}{8}|\sigma(M)| = \frac{11}{8}|16(b-a)| \leq \frac{11}{8}16(b+a) = 22(a+b) \leq b_2(M)$$

as desired. Conversely, suppose that M satisfies the $11/8$-conjecture. Since M has even intersection form and torsion-free homology, it is spin by Proposition 3.37. Now a spin 4-manifold has signature divisible by 16, according to Rochlin's theorem[5]. So we can write $\sigma(M) = 16b$ for some $b \in \mathbb{Z}$. There are three cases to consider.

[5] This will appear later in the course, when we discuss the index of the Dirac operator; see Theorem 4.13.

- If $b = 0$, then $\sigma(M) = 0$. In this case the Hasse-Minkowski classification tells us that $Q_M \cong aH$ for some $\mathbb{Z} \ni a \geq 1$. Freedman's theorem 3.15 then guarantees that M is homeomorphic to $a(S^2 \times S^2)$.
- If $b < 0$, then we want to show that M is homeomorphic to $|b|K3\#a(S^2 \times S^2)$, where b is determined by $\sigma(M)$, and $a = (b_2(M) - 22|b|)/2$. For this to make sense, we need to check that $a \geq 0$. First of all, since Q_M is even, b_2 is even (by Hasse-Minkowski), so a is certainly an integer. Then
$$b_2(M) \geq \frac{11}{8}|\sigma(M)| = 22|b| \implies b_2(M) - 22|b| \geq 0 \implies a \geq 0\,.$$
- Finally, if $b > 0$, by similar arguments, M is homeomorphic to $b\overline{K3}\#a(S^2 \times S^2)$, where $a = (b_2(M) - 22b)/2 \geq 0$ (again, by the 11/8-conjecture). This completes the proof.

□

4 The Dirac operator and the Seiberg-Witten equations

We are now finally starting on the main part of this course. We shall first review the half de Rham complex that already appeared in the course last semester, and then set up the Seiberg-Witten equations and discuss the first properties.

4.1 Self-duality and the half-de Rham complex

4.1.1 Hodge decomposition

Let V be an oriented vector space with of dimension 4 equipped with a scalar product $\langle\cdot,\cdot\rangle$, i.e. the structure of a tangent space of an oriented, Riemannian manifold.

Definition 4.1 (Hodge star operator)**.** We define the Hodge Star operator $* : \Lambda^k(V^*) \to \Lambda^{4-k}(V^*)$ by
$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \mathrm{vol}\,,$$
where we use the induced inner product on forms to make sense out of $\langle \alpha, \beta \rangle$.

It satisfies $*^2 = \mathrm{Id}$ on $\Lambda^2 V^*$, so it has eigenvalues ± 1. Hence $\Lambda^2 V^* = \Lambda^2_+ V^* \oplus \Lambda^2_- V^*$.

Definition 4.2. $\Lambda^2_\pm(V^*)$ are the space of *self-dual* (resp. *anti-self-dual*) 2-forms.

Recall that given an oriented orthonormal basis, $\{e_0, \dots e_3\}$, we have the following basis for $\Lambda^2_\pm(V^*)$:
$$e_0 \wedge e_1 \pm e_2 \wedge e_3$$
$$e_0 \wedge e_2 \mp e_1 \wedge e_3$$
$$e_0 \wedge e_3 \pm e_1 \wedge e_2$$

Now let X be an oriented Riemannian 4-manifold with a metric g. We then have the decompositions

$$\begin{aligned} \Lambda^2 T^*X &= \Lambda^2_+ T^*X \oplus \Lambda^2_- T^*X \\ \implies \Omega^2(TX) &= \Omega^2_+(X) \oplus \Omega^2_-(X) \, . \end{aligned}$$

Recall the L^2 inner product of forms (here, we start assuming that X is closed):

$$\langle \alpha, \beta \rangle_{L^2} = \int_X g(\alpha, \beta) \mathrm{vol}_g$$

where we have once again extended $g(-,-)$ to forms.

Definition 4.3 (Laplace operator)**.** We define the Laplace operator of g as $\Delta \coloneqq \mathrm{dd}^* + \mathrm{d}^*\mathrm{d}$, where $\mathrm{d}^* : \Omega^k(X) \to \Omega^{k-1}(X)$. is the formal adjoint of d with respect to the L^2 scalar product, $\mathrm{d}^* = \pm * \ \mathrm{d} \, *$, where the sign depends on the dimension and degree of the form it acts on. A form $\alpha \in H^*(X;\mathbb{Z})$ is called *harmonic* if $\Delta\alpha = 0$ and the space of harmonic k-forms is denoted by $\mathcal{H}^k(X)$.

Lemma 4.4. *If X is closed, then for a 2-form α we have $\Delta\alpha = 0$ if and only if $\mathrm{d}\alpha = 0 = \mathrm{d}^*\alpha$.*

Proof. We simply use that d and d^* are each other's adjoints:

$$\int_X g(\Delta\alpha, \alpha)\mathrm{vol}_g = \int_X \left(|\mathrm{d}\alpha|^2 + |\mathrm{d}^*\alpha|^2 \right) \mathrm{vol}_g \, .$$

This shows the equivalence. □

Hence, every harmonic form α on a closed Riemannian manifold is closed. Therefore there is a canonical mapping $\mathcal{H}^i(X) \to H^i_{\mathrm{dR}}$.

Theorem 4.5 (Hodge)**.** *Every de Rham cohomology class contains a unique harmonic representative, so that $H^k_{dR}(X) \cong \mathcal{H}^k(X)$. The isomorphism is given by the projection $\mathcal{H}^i(X) \to H^i_{dR}(M)$. Moreover, there is an orthogonal decomposition*

$$\Omega^k(X) = \mathrm{d}(\Omega^{k-1}(X)) \oplus \mathcal{H}^k(X) \oplus \mathrm{d}^*(\Omega^{k+1}) \, .$$

Notice that $*$ maps $\mathcal{H}^i(X)$ to $\mathcal{H}^{4-i}(X)$, and as above, we have the decomposition of the harmonic 2-forms into the space of self-dual and anti self-dual harmonic 2-forms:

$$\mathcal{H}^2(X) = \mathcal{H}^2_+(X) \oplus \mathcal{H}^2_-(X) \, .$$

Assume that α is closed and (anti-)self-dual. Then first observe that $\mathrm{d}^*\alpha = \pm{*}\mathrm{d}{*}\alpha = \pm{*}\mathrm{d}\alpha = 0$. Furthermore, we have:

$$Q_X(\alpha, \alpha) = \int_X \alpha \wedge \alpha = \pm \int_X \alpha \wedge *\alpha = \pm \int_X |\alpha|^2 \mathrm{vol}_g \, .$$

Depending on the sign, i.e. on whether α is self-dual or anti-self-dual, this is non-negative or non-positive, with equality if and only if $\alpha \equiv 0$. Therefore $b_2^\pm(X) = \dim \mathcal{H}^2_\pm$. Now consider α self-dual and β anti self-dual. Then

$$\begin{aligned} Q_X(\alpha,\beta) = \int_X \alpha\wedge\beta = \int_X (*\alpha)\wedge\beta = \int_X \beta\wedge *\alpha = \int_X g(\alpha,\beta)\mathrm{vol}_g \\ = -\int_X \alpha\wedge *\beta = -\int_X g(\alpha,\beta)\mathrm{vol}_g \,. \end{aligned}$$

Hence $Q_X(\alpha,\beta) = 0$. Thus, the decomposition $\Omega^2(X) = \Omega^2_+(X)\oplus\Omega^2_-(X)$ is orthogonal with respect to Q_X (which coincides with the L^2-inner product). In fact, the splitting is even orthogonal with respect to the pointwise metric induced by g.

4.1.2 The half-de Rham complex

Proposition 4.6. *For a closed, oriented, smooth 4-manifold X, the following is a complex with finite-dimensional cohomology:*

$$0 \longrightarrow \Omega^0(X) \xrightarrow{\ \mathrm{d}\ } \Omega^1(X) \xrightarrow{\ d^+\ } \Omega^2_+(X) \longrightarrow 0\,.$$

The alternating sum of dimensions of the cohomology is $\frac{1}{2}\big(\chi(X)+\sigma(X)\big)$.

This is sometimes called the half-de Rham complex. The operator d^+ is the composition

$$\Omega^1(X) \xrightarrow{\mathrm{d}} \Omega^2(X) \xrightarrow{\pi^+} \Omega^2_+(X)\,.$$

Proof. Clearly $\mathrm{d}^+\circ\mathrm{d} = \pi^+\circ\mathrm{d}^2 = 0$ and $H^0_{\mathrm{dR}}(X)$ is the first cohomology vector space of the complex. Let $\alpha\in\Omega^1(X)$ lie in the kernel of d^+. Now observe

$$\begin{aligned} 0 = \int_X \mathrm{d}(\alpha\wedge\mathrm{d}\alpha) = \int_X \mathrm{d}\alpha\wedge\mathrm{d}\alpha &= \int_X (\mathrm{d}^+\alpha+\mathrm{d}^-\alpha)\wedge(\mathrm{d}^+\alpha+\mathrm{d}^-\alpha) \\ &= \int_X \mathrm{d}^+\alpha\wedge *\mathrm{d}^+\alpha - \mathrm{d}^-\alpha\wedge *\mathrm{d}^-\alpha \\ &= \int_X (|\mathrm{d}^+\alpha|^2 - |\mathrm{d}^-\alpha|^2)\mathrm{vol}_g\,. \end{aligned}$$

We see that $\mathrm{d}^+\alpha = 0$ if and only if $\mathrm{d}^-\alpha = 0$, which is then equivalent to $\mathrm{d}\alpha = 0$. Hence the middle cohomology vector space is simply $H^1_{\mathrm{dR}}(X)$. For the last, we just need to find $\operatorname{coker}\mathrm{d}^+$. Take $h\in\mathcal{H}^2_+(X)$ and $\alpha\in\Omega^1(X)$. Then

$$\int_X h\wedge\mathrm{d}^+\alpha = \int_X \mathrm{d}^+\alpha\wedge *h = \langle\mathrm{d}^+\alpha, h\rangle_{L^2}$$

but on the other hand

$$\int_X h\wedge\mathrm{d}^+\alpha = \int_X \mathrm{d}(h\wedge\alpha) = 0$$

by Stokes' theorem, plus the fact that $\Omega^2_+(X) \oplus \Omega^2_-(X)$ is an orthogonal decomposition. Hence the image of $\Omega^1(X)$ under d^+ is orthogonal to $\mathcal{H}^2_+(X)$. Now, we use Hodge decomposition (theorem 4.5) to uniquely write $\omega \in \Omega^2_+$ as $\omega = h + \mathrm{d}\alpha + \mathrm{d}^*\beta$ where h is harmonic. By uniqueness of Hodge decomposition and self-duality, we see $*h = h$, $*\mathrm{d}\alpha = \mathrm{d}^*\beta$ and $*\mathrm{d}^*\beta = \mathrm{d}\alpha$, i.e. $\omega = h + \mathrm{d}\alpha + *\mathrm{d}\alpha = h + 2\mathrm{d}^+\alpha$. But then clearly $\Omega^2_+(X)/\mathrm{d}^+(\Omega^1(X)) \cong \mathcal{H}^2_+(X)$, which is our third cohomology vector space. The alternating sum of the dimensions is:

$$b_0(X) - b_1(X) + b_2^+(X) = \frac{1}{2}\chi(X) + \frac{1}{2}b_2^+(X) - \frac{1}{2}b_2^-(X) = \frac{1}{2}(\chi(X) + \sigma(X))$$

as claimed. □

We can "roll up" all the information about this complex into a single invariant. Consider the operator $\mathrm{d}^+ \oplus \mathrm{d}^* : \Omega^1(X) \to \Omega^2_+(X) \oplus \Omega^0(X)$.

Definition 4.7. The *Fredholm index* of $\mathrm{d}^+ \oplus \mathrm{d}^*$ is defined as

$$\begin{aligned}\operatorname{ind}(\mathrm{d}^+ \oplus \mathrm{d}^*) &= \dim\ker(\mathrm{d}^+ \oplus \mathrm{d}^*) - \dim\operatorname{coker}(\mathrm{d}^+ \oplus \mathrm{d}^*) \\ &= b_1(X) - b_2^+(X) - b_0(X)\end{aligned}$$

since $\ker(\mathrm{d}^+ \oplus \mathrm{d}^*) = (\mathcal{H}^1(X) \oplus \mathrm{d}^*\Omega^2(X)) \cap (\mathcal{H}^1(X) \oplus \mathrm{d}\Omega^0(X)) = \mathcal{H}^1(X)$. Here, we used the fact that an operator with finite-dimensional kernel and cokernel is Fredholm to make the index well-defined.

4.2 Elliptic Operators

Let $E,\ F \to M$ be vector bundles, and $P : \Gamma(E) \to \Gamma(F)$ a first-order differential operator. Think of P as expressed through a covariant derivative, using only first derivatives.

Definition 4.8. The *symbol* of P is a bundle map $\sigma(P) : T^*M \to \operatorname{Hom}(E, F)$, defined as follows. Let $\xi \in T^*_pM$, and $e \in E_p$. Choose an extension $\tilde{e}$ of e to a section of E, and choose a smooth function $f \in C^\infty(M)$ on M with $f(p) = 0$ and $(\mathrm{d}f)_p = \xi$ (e.g. multiply a representative of the germ which satisfies these conditions with a cutoff function). Then

$$(\sigma(P)(\xi))(e) := (P(f \cdot \tilde{e}))(p)\ .$$

We will not show here that this is well-defined, but I think I may have done that already last semester.

Definition 4.9 (Elliptic operator)**.** The operator P is *elliptic* if $\sigma(P)(\xi) \in \operatorname{Hom}(E, F)$ is an isomorphism for all $\xi \neq 0$.

If P is elliptic, then, on a closed manifold, it is also *Fredholm*, that is, its kernel and cokernel are both finite-dimensional. In particular, the Fredholm index $\operatorname{ind} P := \dim\ker P - \dim\operatorname{coker} P$ is well-defined. Note that P can only be elliptic if $\operatorname{rank} E = \operatorname{rank} F$.

Example 4.10.

(i) $P = \mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$. As in the definition above, pick an $\omega \in \Lambda^k(T_p^*(M))$, and extend it to a form $\tilde{\omega} \in \Omega^k(M)$. Choose a function f such that $f(p) = 0$ and $(\mathrm{d}f)(p) = \xi$ for a given $\xi \in T_p^* M$. Then

$$\mathrm{d}(f \cdot \tilde{\omega})(p) = (\mathrm{d}f \wedge \tilde{\omega} + f\mathrm{d}\tilde{\omega}(p)\ . = (\xi \wedge \omega)(p)$$

Hence, $\sigma(\mathrm{d})(\xi) = \xi\wedge$. But this map is not invertible, since e.g. multiples of ξ are in its kernel. Hence, d is not an elliptic operator. However, we see that if we take a direct sum of (non-elliptic) operators *whose symbols have non-overlapping kernel*, we can obtain an elliptic operator: This is exactly what we did with $\mathrm{d}^+ \oplus \mathrm{d}^*$.

(ii) $P = D_A : \Gamma(V) \to \Gamma(V)$, the Dirac operator of a Spin^c-structure with Spin^c-connection A. Once again, pick a $\varphi \in V_p$ and extend to $\tilde{\varphi} \in \Gamma(V)$; fix f with $f(p) = 0$ and $(\mathrm{d}f)(p) = \xi$ for a $\xi \in T_p^* V$. Then

$$\begin{aligned} D_A(f \cdot \tilde{\varphi})(p) &= \gamma_{\text{eval}} \circ g(\nabla^A(f\tilde{\varphi})(p)) = \gamma_{\text{eval}} \circ g((\mathrm{d}f \otimes \tilde{\varphi} + f\nabla^A\tilde{\varphi})(p)) \\ &= \gamma(\xi^*) \cdot \varphi \end{aligned}$$

where ξ^* is dual to ξ under the identification $T_p^*M \cong T_pM$ induced by g. Hence, $\sigma(D_A)(\xi) = \gamma(\xi^*)$ and D_A is elliptic (since Clifford multiplication with a fixed element is an isomorphism). Since D_A is formally self-adjoint, one typically finds $\operatorname{ind} D_A = 0$. However, there are ways to "break the symmetry" and obtain something interesting.

We now specialize to a closed, oriented, smooth $4k$-manifold M with Spin structure, $V = V_+ \oplus V_-$ and $D_A^+ : \Gamma(V_+) \to \Gamma(V_-)$, which is elliptic but not self-adjoint. (The full Dirac operator $D_A^+ \oplus D_A^-$ is self-adjoint, but the half-Dirac operator D_A^+ is not.) A celebrated theorem then relates the Fredholm index of D_A^+ to a topological quantity:

Theorem 4.11 (Atiyah-Singer Index Theorem)**.**

$$\operatorname{ind}_{\mathbb{C}} D_A^+ = \langle \hat{A}(M), [M] \rangle$$

where $\hat{A}(M) = 1 - (1/24)p_1(M) + \ldots$.

We have decorated the index with the subscript $\mathbb{C}$ in order to emphasise that the spinor bundles are complex vector bundles, the Dirac operator is complex linear, and therefore its kernel and cokernel are complex vector spaces. The index formula computes the complex index. If we forget the complex structure and think of real dimensions, we need to take twice this number.

On a 4-manifold, only the degree 4 part of the $\hat{A}$-genus is relevant. We obtain

$$\operatorname{ind}_{\mathbb{C}} D_A^+ = -\frac{1}{24} \langle p_1(TM), [M] \rangle = -\frac{1}{8}\sigma(M)\ . \tag{4.1}$$

Corollary 4.12. *If M is* Spin*, $\sigma(M)$ is divisible by 8.*

This is not really new, since the spin condition implies that the intersection form is even, and we have seen already that for even intersection forms the signature is divisible by 8. In fact, one can do better than this:

Theorem 4.13 (Rochlin)**.** *If M is a smooth closed oriented* Spin *manifold of dimension four, then* $\sigma(M) \equiv 0 \mod 16$.

Proof. We use the Atiyah-Singer index formula. The kernel and cokernel of D_A^+ are $\mathbb{C}$-vector spaces with $\mathrm{ind}_{\mathbb{C}} D_A^+ = \dim_{\mathbb{C}} \ker D_A^+ - \dim_{\mathbb{C}} \mathrm{coker}\, D_A^+ = -\frac{1}{8}\sigma(M)$. In the case of a Spin structure, charge conjugation preserves the kernel and cokernel of D_A^+. Hence, these are in fact quaternionic vector spaces, hence their $\mathbb{C}$-dimensions are in fact even. Thus, the difference is even, i.e. $\sigma(M)/8 \equiv 0 \mod 2$. □

Remark 4.14. This shows that, in the intersection form, only even multiples of E_8 may occur. Hence, many simply connected 4-manifolds with even intersection form do not admit a smooth structure since if they did, they would be Spin.

For a Spin^c structure on a closed oriented smooth manifold M^4 which does not necessarily come from a Spin structure there is a generalization of the index formula:

$$\begin{aligned}\mathrm{ind}_{\mathbb{C}} D_A^+ &= \left\langle \hat{A}(M) \cdot e^{c_1(L_{\mathfrak{s}})/2}, [M] \right\rangle \\ &= \left\langle \left(1 - \frac{1}{24}p_1(M) + \dots\right)\left(1 + \frac{1}{2}c_1(L_{\mathfrak{s}}) + \frac{1}{8}c_1^2(L_{\mathfrak{s}}) + \dots\right), [M] \right\rangle \\ &= \frac{1}{8}\left(\langle c_1^2(L_{\mathfrak{s}}), [M]\rangle - \sigma(M)\right) .\end{aligned}$$

4.3 The Weitzenböck formula

Recall that in section 2.5.1, we defined the Dirac operator D on $\mathbb{R}^4$ such that $D^2 = \Delta$. We now wish to make a similar construction on closed 4-manifolds. Let X be a closed oriented smooth 4-manifold with a Spin^c-structure $\mathfrak{s}$, spinor bundle $V = V_+ \oplus V_-$ and characteristic line bundle $L_{\mathfrak{s}} = \det V_{\pm}$. Let $\hat{A}$ be a $U(1)$-connection on $L_{\mathfrak{s}}$. Together with the Levi-Civita connection of g, this yields a Spin^c-connection A on V (cf. corollary 2.66). Recall that in lemma 2.53, we obtained the following expression for the Dirac operator $D_A : \Gamma(V) \to \Gamma(V)$ with respect to a local orthonormal frame $\{e_1, \dots e_4\}$ of (TX, g):

$$D_A\phi = \sum_{i=1}^{4} e_i \cdot \nabla_{e_i}^A \phi .$$

We now state the main result of the section.

Theorem 4.15 (Weitzenböck formula)**.** *The Dirac operator satisfies*

$$D_A^2 = D_A \circ D_A = \nabla_A^* \nabla_A + \frac{1}{4}s_g + \frac{1}{2}\gamma\left(F_{\hat{A}}\right)$$

where $F_{\hat{A}} \in \Omega^2(X;\mathfrak{u}(1))$ is the curvature of $\hat{A}$, $\gamma(F_{\hat{A}}) \in \mathrm{End}(V)$ is the extension of Clifford multiplication to 2-forms and s_g is the scalar curvature of g which acts on V by multiplication.

We split the proof up into a sequence of lemmata.

Definition 4.16 (Bochner Laplacian)**.** The operator $\nabla_A^*\nabla_A : \Gamma(V) \to \Gamma(V)$ is called the Bochner Laplacian. Recall that the covariant derivative $\nabla_A : \Gamma(V) \to \Gamma(T^*X \otimes V)$, so we define its formal adjoint $\nabla_A^* : \Gamma(T^*X \otimes V) \to \Gamma(V)$ as the adjoint with respect to the L^2-inner product: $\langle \nabla_A^*\phi, \psi\rangle_{L^2} = \langle \phi, \nabla_A\psi\rangle_{L^2}$.

Lemma 4.17. *Let $\{e_i\}$ be a local orthonormal basis for X. Then for $\phi \in \Gamma(V)$, we have*

$$\nabla_A^*\nabla_A\phi = \sum_i \left(-\nabla_{e_i}^A\nabla_{e_i}^A\phi + \nabla_{\nabla_{e_i}e_i}^A\phi\right)$$

where ∇ is the Levi-Civita connection.

Proof. Suppose $\psi \in \Gamma(V)$ has compact support on the open set on which the local frame $\{e_i\}$ is defined and otherwise arbitrary. Then $\nabla_A^*\nabla_A\phi$ is characterized by its inner product with such ψ's. We have, after expanding $\nabla_A\phi$ in the given basis:

$$\langle \nabla_A^*\nabla_A\phi, \psi\rangle_{L^2} = \sum_i \int_X \langle \nabla_{e_i}^A\phi, \nabla_{e_i}^A\psi\rangle \mathrm{vol}_g \ .$$

Here, the pointwise metric $\langle -, -\rangle$ is induced by g and the Hermitian metric on V. On the other hand, we may start on the other side of the identity we want to prove and use that ∇^A is compatible with the metric:

$$\int_X \left\langle \sum_i (-\nabla_{e_i}^A\nabla_{e_i}^A\phi + \nabla_{\nabla_{e_i}e_i}^A\phi, \psi \right\rangle \mathrm{vol}_g = \sum_i \int_X \left(-L_{e_i}\langle \nabla_{e_i}^A\phi, \psi\rangle + \langle \nabla_{e_i}^A\phi, \nabla_{e_i}^A\psi\rangle + \langle \nabla_{\nabla_{e_i}e_i}^A\phi, \psi\rangle\right)\mathrm{vol}_g \ .$$

Thus, all we need to show is that the first and last term cancel. Set $\eta(Y) = \langle \nabla_Y^A\phi, \psi\rangle$ to find:

$$\sum_i \int_X (-L_{e_i}\eta(e_i) + \eta(\nabla_{e_i}e_i))\mathrm{vol}_g = -\sum_i \int_X (\nabla_{e_i}\eta)(e_i)\mathrm{vol}_g \ .$$

Now, we expand in a local parallel frame $\{\omega^j\}$ to see that pointwise $\sum_{i,j}(\nabla_{e_i}(\eta_j\omega^j))(e_i) = \sum_i \partial_i\eta_i$. On the other hand, we have the following pointwise calculation:

$$\begin{aligned} *\mathrm{d} * \Big(\sum_j \eta_j\omega^j\Big) &= *\mathrm{d}\Big(\sum_j \eta_j(-1)^j\omega^0 \wedge \cdots \wedge \hat{\omega}^j \wedge \cdots \wedge \omega^3\Big) = \Big(\sum_j \partial_j\eta_j\Big) * \mathrm{vol} \\ &= \sum_j \partial_j\eta_j \ . \end{aligned}$$

On $\eta \in \Omega^1(X)$, we have $\mathrm{d}^*\eta \,\mathrm{vol}_g = -{*}\,\mathrm{d} * \eta \,\mathrm{vol}_g = \mathrm{d} * \eta$ (where we used that $\mathrm{d} * \eta \in \Omega^4(X)$ and hence $*(\mathrm{d} * \eta)\mathrm{vol}_g = \mathrm{d} * \eta$). Thus, we find

$$\sum_i \int_X (-L_{e_i}\eta(e_i) + \eta(\nabla_{e_i} e_i))\mathrm{vol}_g = \int_X \mathrm{d}^*\eta \,\mathrm{vol}_g = \int_X \mathrm{d} * \eta = 0$$

by Stokes' theorem. This proves our assertion. □

We now recall the following definition.

Definition 4.18 (Curvature)**.** For A a Spin^c-connection on V, we its curvature $F_A \in \Omega^2(X, \mathrm{End}(V))$ is

$$F_A(X, Y)\phi = \nabla^A_X \nabla^A_Y \phi - \nabla^A_Y \nabla^A_X \phi - \nabla^A_{[X,Y]}\phi \,.$$

This is tensorial in X, Y, ϕ.

Thus, we have a map

$$\begin{array}{ccccc} \Gamma(\Lambda^2(T^*X) \otimes \mathrm{End}(V)) & \xrightarrow{\gamma \otimes \mathrm{Id}} & \Gamma(\mathrm{End}(V) \otimes \mathrm{End}(V)) & \xrightarrow{\text{comp.}} & \Gamma(\mathrm{End}(V)) \\ F_A & \longmapsto & & & \gamma(F_A) \end{array}$$

where comp. denotes composition of endomorphisms. If $\{e_i\}$ is a local orthonormal basis, with $\{\omega^i\}$ its dual basis, we have the following expressions:

$$F_A = \sum_{i,j} F_A(e_i, e_j)\omega^i \otimes \omega^j = 2\sum_{i<j} F_A(e_i, e_j)\omega^i \wedge \omega^j$$

$$\implies \gamma(F_A)(\phi) = 2\sum_{i<j} \gamma(e_i \wedge e_j) \circ F_A(e_i, e_j)(\phi) = \sum_{i,j} \gamma(e_i \wedge e_j)(F_A(e_i, e_j)(\phi)) \,.$$

Lemma 4.19. *The Dirac operator satisfies*

$$D_A^2 = \nabla_A^* \nabla_A + \frac{1}{2}\gamma(F_A) \,.$$

Note the appearance of F_A instead of the $F_{\hat{A}}$ from the Weitzenböck formula.

Proof. This is a tensorial equation hence we may use a local parallel frame $\{e_j\}$ in p. Then we simply compute:

$$D_A^2\phi = D_A\Big(\sum_i e_j \cdot \nabla^A_{e_j}\phi\Big) = \sum_{i,j} e_i \cdot \nabla^A_{e_i}(e_j \cdot \nabla^A_{e_j}\phi) = \sum_{i,j} e_i \cdot \big(e_j \cdot \nabla^A_{e_i}\nabla^A_{e_j}\phi\big) \,.$$

Splitting this equation into terms $i = j$ and $i \neq j$ and using the defining properties of Clifford multiplication, we find:

$$D_A^2\phi = \sum_i -\nabla^A_{e_i}\nabla^A_{e_i}\phi + \sum_{i<j} \gamma(e_i \wedge e_j) \cdot (\nabla^A_{e_i}\nabla^A_{e_j} - \nabla^A_{e_j}\nabla^A_{e_i})\phi \,.$$

Now use lemma 4.17 and our expression for the curvature (remembering that $\nabla e_i = 0$ and $[e_i, e_j] = 0$) and conclude

$$D_A^2\phi = \nabla_A^*\nabla_A\phi + \frac{1}{2}\gamma(F_A)(\phi) \ .$$

□

The last question that needs to be settled is: How does F_A relate to $F_{\hat{A}}$? Without proof, we claim:

Lemma 4.20. *Locally, the following formula holds with respect to a local frame $\{e_i\}$:*

$$\gamma(F_A) = \gamma(F_{\hat{A}}) + \frac{1}{4}\sum_i \gamma(e_i)\gamma(R(A))$$

where

$$R(A) = \frac{1}{2}\sum_{j,k,l} R_{ijkl}\, e_j \wedge e_k \wedge e_l$$

and $R_{ijkl} = g(R(e_i, e_j)e_k, e_l)$ is the Riemann curvature tensor.

Taking this at face value, we see that

$$\frac{1}{2}\gamma(F_A) = \frac{1}{2}\gamma(F_{\hat{A}}) + \frac{1}{8}\sum_{i,j,k,l}\gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l)R_{ijkl}$$

and we want to show that this last term equals $\frac{1}{4}s_g$. Recall that $R_{iikl} = R_{ijkk} = 0$ and $R_{ijkl} + R_{iklj} + R_{iljk} = 0$ (the first Bianchi identity). The first identity tells us we can simplify the second term to

$$\frac{1}{2}\sum_{\substack{i<j\\k<l}}\gamma(e_i)\gamma(e_j)\gamma(e_k)\gamma(e_l)R_{ijkl}$$

while the Bianchi identity shows that the terms where i, j, k or i, j, l are all pairwise different cancel out. Therefore, the only terms are contribute are those with $i = k$ and $j = l$ so we find that the second term equals

$$\frac{1}{2}\sum_{i<j}\gamma(e_i)\gamma(e_j)\gamma(e_i)\gamma(e_j)R_{ijij} = -\frac{1}{2}\sum_{i<j}R_{ijij} = \frac{1}{2}\sum_{i<j}R_{ijji}$$

where we used the definition of Clifford multiplication. Now recall that $\mathrm{Ric}(X,Y) = \mathrm{tr}(Z \mapsto R(Z,X)Y)$, i.e. $\mathrm{Ric}_{ij} = \sum_k g(R(e_k, e_i)e_j, e_k)$ and $s_g = \sum_i \mathrm{Ric}_{ii} = \sum_{i,j} R_{ijji} = 2\sum_{i<j} R_{ijji}$, we obtain:

$$\frac{1}{2}\gamma(F_A) = \frac{1}{2}\gamma(F_{\hat{A}}) + \frac{1}{4}s_g$$

This completes our proof of the Weitzenböck formula.

If the smooth closed oriented 4-manifold M^4 is Spin, then the characteristic line bundle L is trivial, hence we may take $\hat{A} = \mathrm{d}$, hence $F_{\hat{A}} = 0$ so that

$$D_A^2 = \nabla_A^* \nabla_A + \frac{1}{4} s_g \, .$$

By the Atiyah-Singer index theorem, $\mathrm{ind}_{\mathbb{C}}\, D_A^+ = -\frac{1}{8}\sigma(M)$. Since the Weitzenböck formula holds on $V_\pm$ with $D_A^2 = D_A^\mp D_A^\pm$, combining it with the Atiyah-Singer theorem yields the so-called "Lichnerowicz argument":

Theorem 4.21 (Lichnerowicz)**.** *If a smooth closed oriented* Spin *4-manifold admits a metric g such that $s_g > 0$, then $\sigma(X) = 0$.*

Proof. We will show that $\mathrm{ind}\, D_A^+ = 0$. First, observe that

$$\mathrm{ind}\, D_A^+ = \dim\ker D_A^+ - \dim\operatorname{coker} D_A^+ = \dim\ker D_A^+ - \dim\ker D_A^-$$

since $D_A^\mp$ is the formal adjoint of $D_A^\pm$. Thus, assume $\phi \in \ker D_A^\pm$. Then we find

$$0 = D_A^\pm D_A^\mp \phi = \nabla_A^* \nabla_A \phi + \frac{1}{4} s_g \phi$$

and taking the L^2-inner product with ϕ itself yields

$$\langle \nabla_A \phi, \nabla_A \phi \rangle_{L^2} + \frac{1}{4}\langle s_g \phi, \phi\rangle_{L^2} = \int_X \Big(|\nabla^A \phi|^2 + \frac{1}{4} s_g |\phi|^2 \Big) \mathrm{vol}_g = 0$$

Since both terms are non-negative ($s_g > 0$!), they must vanish individually. The second term then implies that $\phi = 0$, i.e. any element in $\ker D_A^\pm$ vanishes identically. ☐

If we weaken the curvature assumption, the first term must still vanish identically:

Corollary 4.22. *If M admits a metric with non-negative scalar curvature, $\phi \in \ker D_A^\pm$ implies $\nabla^A \phi = 0$, i.e. ϕ must be parallel.*

If there is a non-zero parallel ϕ, then the second term in the above argument shows in fact that the scalar curvature vanishes identically.

Example 4.23.

(i) $S^2 \times S^2$ is Spin and admits a metric with positive scalar curvature, therefore $\sigma(S^2 \times S^2) = 0$.

(ii) $K3$ is Spin has $\sigma(K3) = -16$. This means it does not admit a metric with $s_g > 0$—but in fact $K3$, being a Calabi-Yau manifold, admits a Ricci-flat (hence $s_g \equiv 0$) Kähler metric.

(iii) The above theorem does not apply to manifolds of the form $\mathbb{C}\mathrm{P}^2 \# k \bar{\mathbb{C}\mathrm{P}}^2$ ($k > 1$), which admit a metric with $s_g > 0$ and have nonzero signature, yet do not admit a Spin structure.

4.4 The Seiberg-Witten equations

Let (X, g) be a smooth closed oriented Riemannian 4-manifold with a Spin^c-structure $\mathfrak{s}$, and equipped with a Spin^c-connection A. Let $\Gamma(V_+)$ denote the space of positive spinors, with $\Phi \in \Gamma(V_+)$. Our main object of study in the following are the Seiberg-Witten (SW) equations for the pair (A, Φ).

The first equation is the *Dirac equation* $D_A^+\Phi = 0$. For the second equation, called the *curvature equation*, we recall (cf. lemma 2.9) that γ induces an isomorphism $\bigwedge_+^2 T^*X \otimes \mathbb{C} \cong \mathrm{End}_0(V_+)$, where End_0 denotes the bundle of traceless endomorphisms. Clifford multiplication γ maps real-valued self-dual forms to traceless, skew-Hermitian endomorphisms and imaginary-valued forms to traceless, Hermitian endomorphisms. In particular, $F_{\hat{A}}^+$ (where $\hat{A}$ is the $U(1)$-connection associated to A) corresponds, as an element of $\Omega_+^2(X, i\mathbb{R})$, to a traceless, Hermitian endomorphism under γ.

Now let $\Phi \in \Gamma(V_+)$ be a positive spinor. We define $\Phi \otimes \Phi^\dagger \in \mathrm{End}(V_+)$ by $(\Phi \otimes \Phi^\dagger)(\psi) = \Phi h(\Phi, \psi)$, where $h(-,-)$ is the Hermitian metric (anti-linear in the *first* entry) on $\Gamma(V_+)$. We denote its traceless part by $(\Phi \otimes \Phi^\dagger)_0$. Consider $\Phi = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$ with respect to a frame for V_+. Then

$$\Phi \otimes \Phi^\dagger = \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix}$$

so

$$(\Phi \otimes \Phi^\dagger)_0 = \begin{pmatrix} \frac{1}{2}\left(|a|^2 - |b|^2\right) & a\bar{b} \\ \bar{a}b & \frac{1}{2}\left(|b|^2 - |a|^2\right) \end{pmatrix}$$

is the desired trace-free endomorphism of V_+.

Definition 4.24. We define $\sigma(\Phi, \Phi) \in \Omega_+^2(X, i\mathbb{R})$ through the equation

$$\gamma(\sigma(\Phi, \Phi)) = (\Phi \otimes \Phi^\dagger)_0 \,.$$

Finally, the curvature equation reads $F_{\hat{A}}^+ = \sigma(\Phi, \Phi)$. Thus, the SW equations for (A, Φ) are:

$$\begin{aligned} D_A^+\Phi &= 0 \,, \\ F_{\hat{A}}^+ &= \sigma(\Phi, \Phi) \,. \end{aligned}$$

Because of the physical interpretation of these equations in terms of massless magnetic monopoles, they are sometimes called the *monopole equations*.

As mentioned in the introduction, the monopole equations are nonlinear partial differential equations. The *Dirac equation* may look linear, and indeed the Dirac operator D_A^+ for a fixed connection A is a linear operator which is being applied to Φ. However, we do not consider A fixed, rather it is one of the variables. Moreover, the curvature equation is quadratic in Φ (through

$\sigma(\Phi, \Phi)$), and second order in A. In applications, it is often useful to *perturb* the curvature equation by a self-dual imaginary-valued form $\omega \in \Omega^2_+(X, i\mathbb{R})$. The ω-perturbed SW equations read:

$$D^+_A \Phi = 0 \,,$$
$$F^+_{\hat{A}} = \sigma(\Phi, \Phi) + \omega \,.$$

Definition 4.25 (SW parameter space)**.** The space of parameters for the SW equations on X is

$$\mathcal{P} = \{(g, \omega) \in \mathrm{Met}(X) \times \Omega^2_+(X, i\mathbb{R})\} \,.$$

Note that the space of self-dual forms in which ω lives depends on the metric g.

Definition 4.26 (SW configuration space)**.** The space

$$\mathcal{C}_\mathfrak{s} = \mathcal{A}_\mathfrak{s} \times \Gamma(V_+)$$

where $\mathcal{A}_\mathfrak{s}$ is the space of Spin^c-connections on V compatible with the Levi-Civita connection is called the Seiberg-Witten configuration space.

Note that by Corollary 2.66, we can identify $\mathcal{A}_\mathfrak{s}$ with $\mathcal{A}(L_\mathfrak{s})$, the space of Hermitian connections on $L_\mathfrak{s}$, which is an affine space over the space $\Omega^1(X, i\mathbb{R})$ of imaginary-valued 1-forms over X.

Corollary 4.27. *The SW configuration space $\mathcal{C}_\mathfrak{s}$ is the product of a vector space and an infinite-dimensional affine space.*

Consider the map

$$f_\omega : \mathcal{C}_\mathfrak{s} \longrightarrow i\Omega^2_+(X) \times \Gamma(V_-)$$
$$(A, \Phi) \longmapsto (F^+_{\hat{A}} - \sigma(\Phi, \Phi) - \omega, D^+_A \Phi) \,.$$

Then the *solution space* of the SW equations

$$\mathcal{Z}_\omega = \{(A, \Phi) \in \mathcal{C}_\mathfrak{s} |\ (A, \Phi) \text{ satisfy the } \omega\text{-perturbed SW equations}\}$$

is just the zero-set of f_ω. For this reason we will sometimes refer to f_ω as the Seiberg-Witten map.

A standard question about equations like SW is "Are they variational equations?" I think it was Richard Feynman who proposed the answer that every equation is "variational" in the following tautological way. If you want to see the equation $f(A, \Phi) = 0$ as arising from an energy functional, take as your energy the norm-squared of f, i.e. $||f||^2 = 0$. This is non-negative, and the equation $f = 0$ describes the absolute minima. In our case the norm is the L^2-norm, and Feynman's functional, for the unperturbed case $\omega = 0$, would be

$$\int_X \left(|D^+_A \Phi|^2 + |F^+_{\hat{A}} - \sigma(\Phi, \Phi)|^2 \right) \mathrm{vol}_g \,.$$

Now there is something much more interesting that we can do here. We are going to modify the Feynman suggestion by adding a topological term depending on the Spin^c-structure $\mathfrak{s}$, and a Riemannian term depending on the scalar curvature. As long as the Riemannian metric is fixed, this just shifts the functional by a constant, and the SW equations still describe the absolute minima. However, the new functional leads to additional insights into the existence or non-existence of solutions to the SW equations.

Definition 4.28. The *energy* of a pair $(A, \Phi) \in \mathcal{C}_{\mathfrak{s}}$ is given by

$$E(A, \Phi) = \int_X \left(|D_A^+ \Phi|^2 + |F_{\hat{A}}^+ - \sigma(\Phi, \Phi)|^2 + \frac{1}{8} s_g^2 \right) \mathrm{vol}_g - 4\pi^2 \langle c_1^2(L_{\mathfrak{s}}), [X] \rangle \,.$$

The following proposition shows that this quantity is always non-negative, although the sign of the topological shift involving the fist Chern class of the Spin^c-structure is not controlled *a priori*, and one could have expected that it destroys non-negativity of the energy.

Proposition 4.29. *The energy can be expressed as:*

$$E(A, \Phi) = \int_X \left(|\nabla^A \Phi|^2 + \frac{1}{8} \big(s_g + |\Phi|^2 \big)^2 + |F_{\hat{A}}^-|^2 \right) \mathrm{vol}_g$$

for all $(A, \Phi) \in \mathcal{C}_{\mathfrak{s}}$.

To prove this, we need some identities:

Lemma 4.30. *Equip* $\mathrm{End}(V_+)$ *with the inner product* $\langle A, B \rangle = \mathrm{tr}(AB^\dagger)$. *Then, for every* $\omega, \eta \in i\Lambda_+^2 T^*X$ *and* $\Phi \in V_+$, *we have:*

(i) $\langle \gamma(\omega), \gamma(\eta) \rangle = 4 \langle \omega, \eta \rangle$.

(ii) $\langle \gamma(\omega)\Phi, \Phi \rangle = 4 \langle \omega, \sigma(\Phi, \Phi) \rangle$.

(iii) $|\Phi|^4 = 8 \langle \sigma(\Phi, \Phi), \sigma(\Phi, \Phi) \rangle$.

Proof. This is left as an exercise. See Ex. 3 on Sheet 6. □

Proof of Proposition 4.29. Recall the Weitzenböck formula, which implies

$$\int_X |D_A^+ \Phi|^2 \mathrm{vol}_g = \int_X \langle D_A^- D_A^+ \Phi, \Phi \rangle \mathrm{vol}_g = \int_X \left(|\nabla^A \Phi|^2 + \frac{1}{4} s_g |\Phi|^2 + \frac{1}{2} \langle \gamma(F_{\hat{A}}^+)\Phi, \Phi \rangle \right) \mathrm{vol}_g \,.$$

Using the lemma, we have $\frac{1}{2} \langle \gamma(F_{\hat{A}}^+ \Phi, \Phi \rangle = 2 \langle F_{\hat{A}}^+, \sigma(\Phi, \Phi) \rangle$. This term is canceled by the second term from the following:

$$\int_X |F_{\hat{A}}^+ - \sigma(\Phi, \Phi)|^2 \mathrm{vol}_g = \int_X \big(|F_{\hat{A}}^+|^2 - 2 \langle F_{\hat{A}}^+, \sigma(\Phi, \Phi) \rangle + |\sigma(\Phi, \Phi)|^2 \big) \mathrm{vol}_g \,.$$

Moreover, $|\sigma(\Phi,\Phi)|^2 = \frac{1}{8}|\Phi|^4$. Putting this term together with the term $\frac{1}{8}s_g^2$ and the scalar curvature term from $|D_A^+\Phi|^2$, we obtain

$$\frac{1}{8}\int_X (s_g + |\Phi|^2)^2 \mathrm{vol}_g \ .$$

Now we have arrived at

$$|D_A^+\Phi|^2 + |F_{\hat{A}}^+ - \sigma(\Phi,\Phi)|^2 + \frac{1}{8}s_g^2 = |\nabla^A\Phi|^2 + \frac{1}{8}\big(s_g + |\Phi|^2\big)^2 + |F_{\hat{A}}^+|^2 \ .$$

Thus, all that is left is to show that

$$-4\pi^2\langle c_1^2(L_{\mathfrak{s}}), [X]\rangle = \int_X \big(|F_{\hat{A}}^-|^2 - |F_{\hat{A}}^+|^2\big)\mathrm{vol}_g \ .$$

This follows from *Chern-Weil theory*, which yields a formula for Chern classes in terms of curvature[6]. In particular, we have $c_1(L_{\mathfrak{s}}) = \frac{i}{2\pi}[F_{\hat{A}}]$. Thus, we find

$$-4\pi^2\langle c_1^2(L_{\mathfrak{s}}), [X]\rangle = \int_X F_{\hat{A}} \wedge F_{\hat{A}} = \int_X F_{\hat{A}}^+ \wedge F_{\hat{A}}^+ + F_{\hat{A}}^- \wedge F_{\hat{A}}^- = \int_X F_{\hat{A}}^+ \wedge (*F_{\hat{A}}^+) - F_{\hat{A}}^- \wedge (*F_{\hat{A}}^-) \ .$$

Recall that the extension of the Hodge star operator to $\mathbb{C}$-valued forms is

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta\rangle \mathrm{vol}_g \ .$$

Since $F_{\hat{A}}^\pm$ takes values in $i\mathbb{R}$, $\bar{F}_{\hat{A}}^\pm = -F_{\hat{A}}^\pm$ and we conclude:

$$-4\pi^2\langle c_1^2(L_{\mathfrak{s}}), [X]\rangle = \int_X \big(-|F_{\hat{A}}^+|^2 + |F_{\hat{A}}^-|^2\big)\mathrm{vol}_g \ .$$

This completes the proof. □

Corollary 4.31. *If there is a solution (A,Φ) to the (unperturbed) SW equations for the Riemannian metric g, then*

$$\langle c_1^2(L_{\mathfrak{s}}), [X]\rangle \leq \frac{1}{32\pi^2}\int_X s_g^2 \mathrm{vol}_g \ .$$

If equality holds, then every solution (A,Φ) has $E(A,\Phi) = 0$, and thus $\nabla^A\Phi = F_{\hat{A}}^- = s_g + |\Phi|^2 = 0$.

Proof. If (A,Φ) solve the SW equations, the definition of the energy becomes

$$E(A,\Phi) = \frac{1}{8}\int_X s_g^2 \mathrm{vol}_g - 4\pi^2\langle c_1^2(L_{\mathfrak{s}}), [X]\rangle \ .$$

By the Proposition this is non-negative and rearranging gives the desired inequality. In the case of equality the formula for the energy in the Proposition gives the second assertion since the non-negative terms in the integrand have to vanish individually. □

[6]This was how we introduced the Euler class last semester, and here the first Chern class is just that Euler class.

4.5 Symmetries of the equations

4.5.1 Charge conjugation

Recall the charge conjugation map $J : \bar{\mathfrak{s}} \to \mathfrak{s}$ from section 2.2.3. It induces a map $\tau : \mathcal{C}_{\bar{\mathfrak{s}}} \to \mathcal{C}_{\mathfrak{s}}$. Let A be a Spin^c-connection on $\bar{V}$. It induces a Spin^c-connection A^* on V, defined by $\nabla_X^{A^*}(J\Phi) = J\nabla_X^A\Phi$ for every $\Phi \in \Gamma\left(\bar{V}\right)$. Now we define τ by

$$\begin{aligned} \tau : \mathcal{C}_{\bar{\mathfrak{s}}} &\longrightarrow \mathcal{C}_{\mathfrak{s}} \\ (A, \Phi) &\longmapsto (A^*, J\Phi) \ . \end{aligned}$$

This satisfies $\tau^2 = \mathrm{Id}$, i.e. τ is an involution.

Lemma 4.32. *A pair $(A, \Phi) \in \mathcal{C}_{\bar{\mathfrak{s}}}$ satisfies the SW equations for parameters (g, ω) if and only if $\tau(A, \Phi) \in \mathcal{C}_{\mathfrak{s}}$ satisfies them for $(g, -\omega)$.*

Proof. We first check the Dirac equation, using that J commutes with γ:

$$D^+_{A^*} J\Phi = \sum_i e_i \cdot \nabla^{A^*}_{e_i} J\Phi = \sum_i e_i \cdot J\nabla^A_{e_i}\Phi = J\sum_i e_i \cdot \nabla^A_{e_i}\Phi = JD^+_A\Phi \ .$$

Thus, (A, Φ) satisfies the Dirac equation if and only if $(A^*, J\Phi)$ does. The curvature satisfies $F_{\hat{A}^*} = -F_{\hat{A}}$ since $\bar{V} \cong V^*$ and the curvature of the dual connection on the dual bundle is the negative the original curvature. Furthermore, an easy exercise (see Ex. 4 on Sheet 6) shows that $\sigma(J\Phi, J\Phi) = -\sigma(\Phi, \Phi)$. Thus, the curvature equation is satisfied by $(A^*, J\Phi)$ if we map ω to $-\omega$. □

4.5.2 Gauge symmetry

The SW equations are invariant under the action of a suitable gauge group. Since our Spin^c-connections are determined to a large extent by the Levi-Civita connection of g, the freedom one has in choosing A is restricted to the choice of a $U(1)$-connection on the determinant bundle of the Spin^c-structure. Because of this, the relevant gauge group is not a group of principal bundle isomorphisms for a frame bundle with non-Abelian structure group, but rather it is the automorphism group of a principal $U(1)$- or S^1-bundle. Because the circle group is Abelian, the relevant bundle of groups is trivial, and its space of sections consists of maps from the base to the circle.

So the *gauge group* we consider is just $\mathcal{G} = C^\infty(X, S^1)$. This acts on the domain and target spaces of the Seiberg-Witten map as follows.

- On $\mathcal{C}_{\mathfrak{s}}$ we have, for $u \in \mathcal{G}$,

  $$(A, \Phi) \mapsto (A, \Phi) \cdot u := ((u^{-1})^* A, u\Phi) \ ,$$

  and the Spin^c-connection transforms as $\nabla^{(u^{-1})^*A} := u\nabla_A u^{-1}$.

- On $i\Omega^2_+(X) \times \Gamma(V_-) \supseteq f_\omega(\mathcal{C}_{\mathfrak{s}})$, we define an action by $(\eta, \psi) \mapsto (\eta, \psi) \cdot u := (\eta, u\psi)$.

Lemma 4.33. *f_ω is equivariant with respect to these actions of $\mathcal{G}$, i.e. $f_\omega((A,\Phi)\cdot v) = f_\omega(A,\Phi)\cdot v$.*

Proof. We can simply write out

$$f_\omega((A,\Phi)\cdot v) = f_\omega((v^{-1})^*A, v\Phi) = (F^+_{\widehat{(v^{-1})^*A}} - \sigma(v\Phi, v\Phi) - \omega, D^+_{(v^{-1})^*A}(v\Phi))\ .$$

Firstly, $\sigma(v\Phi, v\Phi) = \sigma(\Phi,\Phi)$, since by definition of σ, we have

$$\gamma(\sigma(\Phi,\Phi)) = (\Phi\otimes\Phi^\dagger)_0 = (v\Phi\otimes\bar{v}\Phi^\dagger)_0 = \gamma(\sigma(v\Phi, v\Phi))$$

where we used that v takes values in S^1, i.e. $\bar{v} = v^{-1}$. Next, we consider the curvature, using $\nabla^{\widehat{(v^{-1})^*A}}\Phi = v\cdot\nabla^A(v^{-1}\cdot\Phi)$ where $\cdot$ denotes the corresponding group action (which we need not explicitly know) and will be omitted in the following:

$$\begin{aligned}
F^+_{\widehat{(v^{-1})^*A}}(X,Y)s &= \big([\nabla_X^{\widehat{(v^{-1})^*A}}, \nabla_Y^{\widehat{(v^{-1})^*A}}] - \nabla_{[X,Y]}^{\widehat{(v^{-1})^*A}}\big)s \\
&= v[\nabla_X,\nabla_Y]v^{-1}s - v\nabla_{[X,Y]}v^{-1}s \\
&= v\nabla_X((L_Y v^{-1})s + v^{-1}\nabla_Y s + \dots \\
&= v(L_X L_Y v^{-1})s + v(L_Y v^{-1})\nabla_X s + v(L_X v^{-1})\nabla_Y s + \nabla_X\nabla_Y s + \dots
\end{aligned}$$

Note that the middle terms are symmetric in X, Y, hence will disappear. The first term will cancel against one term of $v(L_{[X,Y]}v^{-1})$, so only the last term remains. This shows that we recover $F^+_{\hat{A}}(X,Y)s$.

Finally, we check that $D^+_{(v^{-1})^*A}v\Phi = vD^+_A\Phi$. But this is immediate:

$$D^+_{(v^{-1})^*A}v\Phi = \sum_i e_i\cdot\nabla_{e_i}^{(v^{-1})^*A}v\Phi = \sum_i e_i\cdot v\nabla^A_{e_i}\Phi = vD^+_A\Phi$$

completing our proof that $f_\omega((A,\Phi)\cdot v) = (F^+_{\hat{A}} - \sigma(\Phi,\Phi) - \omega, vD^+_A\Phi) = f_\omega(A,\Phi)\cdot v$. □

We would like to know whether the gauge group action on the configuration space is free, and, if not, then we want to determine what the possible stabilizers are.

Lemma 4.34. *If X^4 is connected, the stabilizer, $\mathcal{G}_{(A,\Phi)}$ of $(A,\Phi)\in\mathcal{C}_\mathfrak{s}$ is given by*

$$\mathcal{G}_{(A,\Phi)} = \begin{cases}\{1\} & \text{if } \Phi\not\equiv 0\ ,\\ \mathrm{U}(1) & \text{if } \Phi\equiv 0\ .\end{cases}$$

Proof. That u is contained in the stabilizer of (A,Φ) means $(A,\Phi)\cdot u = (A,\Phi)$, i.e. $u\nabla^A u^{-1} = \nabla^A$ and $u\Phi = \Phi$. Since $u\nabla^A u^{-1} = \nabla^A + u\mathrm{d}(u^{-1})$, the former equation holds if and only if $u\mathrm{d}(u^{-1}) = -u^{-1}\mathrm{d}u = 0$, hence $u\in S^1\cong\mathrm{U}(1)$ is constant. If $\Phi\equiv 0$, the second condition is trivial and every constant $u\in\mathrm{U}(1)$ is an element of the stabilizer. If Φ does not vanish identically, then a constant gauge transformation acts non-trivially on it by rotation, so the stabilizer is trivial. □

So the action is indeed free almost everywhere. Only on the subset of infinite codimension where the spinor vanishes identically, there is a non-trivial stabilizer. Recall that for connections on principal bundles with a structure group with trivial center, a non-trivial stabilizer in the gauge group means that the connection is reducible to a proper subgroup. We transplant this terminology of reducible and irreducible connections to the pairs (A, Φ) which are the configurations for the SW equations. Correspondingly, we make the following definition:

Definition 4.35. The space of *irreducible configurations* is

$$\mathcal{C}^*_{\mathfrak{s}} = \{(A, \Phi) \in \mathcal{C}_{\mathfrak{s}} \mid \Phi \not\equiv 0\} .$$

The complement $\mathcal{C}_{\mathfrak{s}} \setminus \mathcal{C}^*_{\mathfrak{s}}$ is the set of *reducible* configurations.

4.6 Analytical setup

So far we have worked with configuration spaces and gauge groups made up of smooth objects, that is smooth connections, smooth sections of spinor bundles, smooth maps from the base manifold to the circle, and so forth. These spaces of smooth objects are only Frechet manifolds and are not convenient for doing analysis. In order to study the structure of the solution space to the SW equations near a given solution we want to implement a two-step strategy, where we first look at the linearized equations, and then consider the remaining non-linear equations and use an implicit function theorem. In order to be able to apply a standard implicit function theorem we need a setup with Banach manifolds, that is infinite-dimensional manifolds modelled on Banach spaces. The best framework for this is supplied by the machinery of Sobolev spaces.

Consider a vector bundle $E \to X$ with a Hermitian metric. On smooth sections $\Gamma(E)$, define a norm

$$\|s\|^p_k := \left(\int_X \left(|s|^p + |\nabla s|^p + \ldots + |\nabla^k s|^p\right) \mathrm{vol}_g \right)^{1/p}$$

for $p,\ k \in \mathbb{N}$.

Definition 4.36 (Sobolev space)**.** The Banach space completion of $\Gamma(E)$ with respect to $\|\cdot\|^p_k$ is a Sobolev space of E, denoted $L^p_k(E)$. We write $L^p(E)$ for $L^p_0(E)$.

In our discussion, we will make the following assumptions on the quantities that appear in our study of the SW equations[7]:

- positive spinors $\Phi \in \Gamma(V_+)$ lie in $L^2_5(V_+)$,
- sections $i\Lambda^2_+(X) \times V_-$ are elements of $L^2_4(i\Lambda^2_+(X) \times V_-)$,

[7]The last two items in this list are not covered by our definition, since these spaces are not those of sections of a vector bundle. However, there are ways to extend the Sobolev space construction to these spaces also: in the case of $\mathcal{G}$, for instance, one can obtain the Sobolev space of sections of the line bundle $\mathbb{C} \to X$ and then just restrict to sections with unit length.

- $\mathcal{A}_\mathfrak{s} \in L^2_5(\mathcal{A})$, i.e. of the form $\hat{A}_0 + a$ for $\hat{A}_0$ a smooth connection on $L_\mathfrak{s}$ and $a \in iL^2_5(T^*X)$,
- $\mathcal{G}$ consists of maps in $L^2_6(X, S^1)$.

We will slightly abuse notation and keep using the old symbols when we are actually referring to the corresponding Sobolev spaces, rather than the Frechet spaces made up of smooth objects.

Our choice of Sobolev spaces is made so that all objects in these Sobolev spaces are continuous by the Sobolev embedding theorem and so that the following proof works. We could just as well choose L^2_k spaces with larger k as long as in the gauge group we have control on one more derivative than the configuration space, and in the configuration spaces we have control on one more derivative than in the target of the Seiberg-Witten map, because of the loss of one degree of regularity under differentiation.

Lemma 4.37. *The L^2_6-gauge group $\mathcal{G}$ is an infinite-dimensional Abelian Hilbert Lie group*[8]*. It acts smoothly on the L^2_5-configuration space $\mathcal{C}_\mathfrak{s}$, and on L^2_4-sections of $i\Lambda^2_+(X) \times V_-$.*

Proof. First notice that the multiplication mapping $L^2_6(X, \mathbb{C}) \times L^2_6(X, \mathbb{C}) \to L^2_6(X, \mathbb{C})$ is continuous by the Sobolev multiplication theorem and it is easy to check smoothness. Next observe that $\mathcal{G}$ is the preimage of $1 \in L^2_6(X, \mathbb{R})$ under the (smooth) map $u \to u\bar{u}$ and check that 1 is a regular value.

It is clear that with this manifold structure on $\mathcal{G}$ taking inverses and multiplication are smooth maps being the restriction of smooth maps on $L^2_6(X, \mathbb{C})$.

Again by Sobolev multiplication, we have continuous maps

$$L^2_6 \times L^2_5 \longrightarrow L^2_5$$

and

$$L^2_6 \times L^2_4 \longrightarrow L^2_4$$

so that the L^2_6 gauge group does indeed act on the completed domains and targets for the Seiberg-Witten map. $\square$

4.7 The linearized equations

With our choice of Sobolev completions, the Seiberg-Witten map $f_\omega : \mathcal{C}_\mathfrak{s} \to i\Omega^2_+ \times \Gamma(V_-)$ becomes a smooth map between Banach manifolds. Its derivative is given by the *linearization* of the SW equations.

Lemma 4.38. *For $\omega \in iL^2_4(\Lambda^2_+ T^*X)$ and (A, Φ) in the L^2_5-configuration space $\mathcal{C}_\mathfrak{s}$, the differential of f_ω is given by:*

$$\begin{aligned} \mathcal{T}_{(A,\Phi)} f_\omega : i\Omega^1(X) \times \Gamma(V_+) &\longrightarrow i\Omega^2_+(X) \times \Gamma(V_-) \\ (a, \varphi) &\longmapsto \left(2\mathrm{d}^+ a - \sigma(\Phi, \varphi) - \sigma(\varphi, \Phi), D^+_A \varphi + \gamma(a)\Phi\right) . \end{aligned}$$

[8] See e.g. here for some more information.

Proof. The domain and codomain are correct, since $\mathcal{A}_\mathfrak{s}$ is an affine space over $i\Omega^1(X)$ while $\Gamma(V_+)$ is a vector space, as is the target space. Now consider a curve $(A+ta,\Phi)$ through (A,Φ). Then we find

$$\mathcal{T}_{(A,\Phi)}f_\omega(a,0) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} f_\omega(A+ta,\Phi) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} (F^+_{\widehat{A+ta}} - \sigma(\Phi,\Phi) - \omega, D^+_{A+ta}\Phi)\ .$$

We have $F^+_{\widehat{A+ta}} = F^+_{\hat{A}} + 2t\mathrm{d}^+a$, with the factor of 2 appearing because $\hat{A}$ is defined on the determinant bundle of the rank two bundle V_+. Furthermore,

$$D^+_{A+ta}\Phi = \sum_i e_i \cdot \left(\nabla^A_{e_i} + ta(e_i)\right)\Phi\ . = D^+_A\Phi + t\gamma(a)\Phi$$

Thus we conclude that

$$\mathcal{T}_{(A,\Phi)}f_\omega(a,0) = (2\mathrm{d}^+a, \gamma(a)\Phi)\ .$$

Proceeding in the same way it is not hard to show

$$\mathcal{T}_{(A,\Phi)}f_\omega(0,\varphi) = (-\sigma(\Phi,\varphi) - \sigma(\varphi,\Phi), D^+_A\varphi)\ .$$

Putting these results together completes the proof. □

Let us now examine the infinitesimal or linearized action induced by $\mathcal{G}$ on $\mathcal{C}_\mathfrak{s}$.

Lemma 4.39. *Fix $(A,\Phi)\in\mathcal{C}_\mathfrak{s}$. The action of $\mathcal{G}$ on $\mathcal{C}_\mathfrak{s}$ induces a map*

$$\begin{aligned}\mathfrak{g} = i\Omega^0(X) &\xrightarrow{L_{(A,\Phi)}} T_{(A,\Phi)}\mathcal{C}_\mathfrak{s} = i\Omega^1(X)\times\Gamma(V_+)\\ \xi &\longmapsto (-\mathrm{d}\xi, \xi\Phi)\end{aligned}$$

where $\mathfrak{g}$ is the Lie algebra of $\mathcal{G}$, and $T_{(A,\Phi)}\mathcal{C}_\mathfrak{s}$ is the tangent space of $\mathcal{C}_\mathfrak{s}$ at (A,Φ).

Proof. We use the fact that $\exp(t\xi)$ has tangent vector ξ at $t=0$ to compute:

$$\begin{aligned}L_{(A,\Phi)}\xi &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(A,\Phi)\cdot\exp(t\xi) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0}(A+\exp(t\xi)\mathrm{d}(\exp(-t\xi)), \exp(t\xi)\Phi)\\ &= (-\mathrm{d}\xi, \xi\Phi)\ .\end{aligned}$$

□

This leads into the following result.

Proposition 4.40. *For fixed (A,Φ), we consider the composition*

$$i\Omega^0(X) \xrightarrow{L_{(A,\Phi)}} i\Omega^1\times\Gamma(V_+) \xrightarrow{\mathcal{T}_{(A,\Phi)}f_\omega} i\Omega^2_+(X)\times\Gamma(V_-)\ .$$

If $D^+_A\Phi = 0$, then for all $\omega\in i\Omega^2_+(X)$, the above is an elliptic complex with index (i.e. Euler characteristic[9]) equal to $-\frac{1}{4}\left(c_1^2(L_\mathfrak{s}) - (2\chi(X)+3\sigma(X))\right)$.

[9] As usual, the Euler characteristic is defined to be the alternating sum of the dimensions of cohomology vector spaces.

Proof. To show that we are dealing with a complex, we must show $\mathcal{T}_{(A,\Phi)} f_\omega \circ L_{(A,\Phi} = 0$. Using our previous lemmata, we find for $\xi \in i\Omega^0(X)$:

$$\mathcal{T}_{(A,\Phi)} f_\omega \circ L_{(A,\Phi)}(\xi) = \mathcal{T}_{(A,\Phi)} f_\omega(-\mathrm{d}\xi, \xi\Phi) = (-2\mathrm{d}^+\mathrm{d}\xi - \sigma(\Phi, \xi\Phi) - \sigma(\xi\Phi, \Phi), D_A^+(\xi\Phi) - \gamma(\mathrm{d}\xi)\Phi) .$$

Clearly $\mathrm{d}^+\mathrm{d}\xi = 0$ while

$$\sigma(\Phi, \xi\Phi) + \sigma(\xi\Phi, \Phi) = \gamma^{-1}((\Phi \otimes (\xi\Phi)^\dagger)_0) + \gamma^{-1}(((\xi\Phi) \otimes \Phi^\dagger)_0) = -\xi\sigma(\Phi, \Phi) + \xi\sigma(\Phi, \Phi)$$

since $\bar{\xi} = -\xi$. Moreover,

$$D_A^+(\xi\Phi) = \sum_i e_i \cdot ((L_{e_i}\xi)\Phi + \xi\nabla_{e_i}\Phi) = \sum_i (e_i \cdot L_{e_i}\xi)\Phi + \xi D_A^+\Phi = \gamma(\mathrm{d}\xi)\Phi + \xi D_A^+\Phi .$$

Thus, if $D_A^+\Phi = 0$ we have a complex, since the first term cancels.

To check ellipticity and compute the index, we need the symbol of these differential operators. Recall from the discussion of the symbol (cf. section 4.2) that it depends only on the terms of highest order, in terms of number of derivatives taken. Here, if we drop the lower order terms, then we get:

$$\begin{array}{ccccc} i\Omega^0(X) & \longrightarrow & i\Omega^1(X) \times \Gamma(V_+) & \longrightarrow & i\Omega^2_+ \times \Gamma(V_-) \\ \xi & \longmapsto & (-\mathrm{d}\xi, 0) & & \\ & & (a, \varphi) & \longmapsto & (2\mathrm{d}^+ a, D_A^+\varphi) . \end{array}$$

Thus, after dropping terms of lower order, which do not affect the symbol, our complex decouples into the direct sum of the following two complexes:

$$i\Omega^0(X) \xrightarrow{-\mathrm{d}} i\Omega^1(X) \xrightarrow{\mathrm{d}^+} i\Omega^2_+(X)$$

$$0 \longrightarrow \Gamma(V_+) \xrightarrow{D_A^+} \Gamma(V_-) .$$

The first of these is the half de Rham complex, which we already know to be elliptic. The second one is just the (half) Dirac operator, which is also elliptic. This proves ellipticity of our complex. To compute the index we simply add the indices of the two complexes whose direct sum we take. The index of the half de Rham complex is $\frac{1}{2}(\chi(X) + \sigma(X))$.

The second complex has index $-\operatorname{ind}_{\mathbb{R}} D_A^+ = -2\operatorname{ind}_{\mathbb{C}} D_A^+$ (since D_A^+ is the *second* map). The Atiyah-Singer index theorem tells us that $\operatorname{ind}_{\mathbb{C}} D_A^+ = \frac{1}{8}(c_1^2(L_\mathfrak{s}) - \sigma(X))$. Putting our results together, we find total (real) index

$$\frac{1}{2}(\chi(X) + \sigma(X)) - \frac{1}{4}(c_1^2(L_\mathfrak{s}) - \sigma(X)) = \frac{1}{4}(2\chi(X) + 3\sigma(X) - c_1^2(L_\mathfrak{s})) .$$

as claimed. □

Remark 4.41. It is a general fact that

$$\frac{1}{4}\left(c_1^2(L_{\mathfrak{s}}) - (2\chi(X) + 3\sigma(X))\right) = c_2(V_+)\ .$$

This follows from

$$p_1(\Lambda^2_+(X)) = 2\chi(X) + 3\sigma(X) = c_1^2(V_+) - 4c_2(V_+) = c_1^2(L_{\mathfrak{s}}) - 4c_2(V_+)\ .$$

This is discussed in Exercise 4 on Sheet 7.

5 Moduli spaces of monopoles

We now want to study the solution spaces of the Seiberg-Witten or monopole equations.

We consider a smooth closed oriented 4-manifold X equipped with a fixed Spin^c-structure $\mathfrak{s}$. For parameters $(g, \omega) \in \mathcal{P}$ the Seiberg-Witten equations define a map

$$\begin{aligned} f_\omega : \mathcal{C}_{\mathfrak{s}} &\longrightarrow i\Omega^2_+(X) \times \Gamma(V_-) \\ (A, \Phi) &\longmapsto (F^+_{\hat{A}} - \sigma(\Phi, \Phi) - \omega, D^+_A \Phi) \end{aligned}$$

so that the *solution space* of the SW equations

$$\mathcal{Z}_\omega = \{(A, \Phi) \in \mathcal{C}_{\mathfrak{s}} |\ (A, \Phi) \text{ satisfy the } \omega\text{-perturbed SW equations}\}$$

is just the zero-set of f_ω. Because the map f_ω is equivariant for the action of the gauge group $\mathcal{G}$, the gauge group preserves the zero-set $\mathcal{Z}_\omega$ and acts on it.

Definition 5.1 (SW moduli spaces)**.** The moduli space of gauge equivalence classes of solutions to the ω-perturbed Seiberg-Witten equations is $\mathcal{M}_\omega := \mathcal{Z}_\omega/\mathcal{G} \subset \mathcal{B}$. Here $\mathcal{B} := \mathcal{C}_{\mathfrak{s}}/\mathcal{G}$ the quotient of the configuration space by the gauge group action.

For all these spaces there are starred versions, e.g. $\mathcal{M}^*_\omega \subset \mathcal{B}^*$, denoting the subsets of irreducible configurations, that is, those where Φ does not vanish identically.

Throughout this chapter we fix the analytical setup specified in section 4.6. This implies that $\mathcal{C}_{\mathfrak{s}}$ is a Banach manifold on which the Banach Lie group $\mathcal{G}$ acts smoothly. The quotient $\mathcal{B}$ is not quite a Banach manifold, since the action is not free. We now start to analyze the structure of $\mathcal{G}$ and $\mathcal{B}$, and this will eventually allow us to get a grip on the structure of the moduli space $\mathcal{M}_\omega$.

5.1 The structure of the gauge group

Recall that the gauge group $\mathcal{G}$ consists of maps $u\colon X \longrightarrow S^1$. Initially we took smooth maps, but then completed in the L^2_6 Sobolev norm. By the Sobolev embedding theorem all these maps are continuous, and this allows us to apply the usual discussion of homotopy classes of continuous maps to the elements of $\mathcal{G}$.

Definition 5.2. We define the *degree* of a gauge transformation by the map

$$\begin{aligned} \deg : \quad \mathcal{G} &\longrightarrow H^1(X;\mathbb{Z}) \\ u &\longmapsto u^*\mu \ , \end{aligned}$$

where μ is a fixed generator of $H^1(S^1;\mathbb{Z})$.

Proposition 5.3. *The set of homotopy classes $[X, S^1]$ equipped with the group structure induced by pointwise multiplication in S^1 is naturally isomorphic to $H^1(X;\mathbb{Z})$.*

Proof. Homotopy classes of maps $X \to S^1$ are in bijection with homomorphisms $\pi_1(X) \to \mathbb{Z}$, but since $\mathbb{Z}$ is Abelian and $H_1(X)$ is the Abelianization of $\pi_1(X)$, any such homomorphism factors through $H_1(X)$. Hence $[X, S^1] = \mathrm{Hom}(H_1(X), \mathbb{Z}) = H^1(X;\mathbb{Z})$. □

Corollary 5.4. $\deg u = 0$ *if and only if* $u \simeq$ *const.*

Definition 5.5. We denote by

$$\mathcal{G}_0 = \ker \deg = \{u \in \mathcal{G} |\, \deg u = 0 \Leftrightarrow u \simeq \text{const.}\}$$

the subgroup of null-homotopic elements of $\mathcal{G}$.

The following is an elementary result from covering space theory:

Lemma 5.6. $\deg u = 0$ *if and only if* $u = e^{if}$ *with* $f : X \to \mathbb{R}$ *globally defined.*

Proof. A continuous map $F : X \to S^1$ lifts to $\mathbb{R}$, i.e. $F = e^{if}$ for $f : X \to \mathbb{R}$, if and only if $F_*\pi_1(X)$ is trivial in $\pi_1(S^1) \cong H_1(S^1;\mathbb{Z})$, i.e. $\deg F = 0$. □

Corollary 5.7. *The subgroup $\mathcal{G}_0$ is the connected component of the neutral element in $\mathcal{G}$ considered as a topological group, and $\mathcal{G}/\mathcal{G}_0 \cong H^1(X;\mathbb{Z}) = \mathbb{Z}^{b_1(X)}$ with the isomorphism induced by the degree map.*

Consider the following two subgroups of $\mathcal{G}_0$:

- $\mathrm{U}(1) = \{e^{ic} \mid c \in \mathbb{R}\}$.
- $\mathcal{G}^\perp := \{e^{if} \mid f \in L^2_6(X),\ \int_X f d\mathrm{vol}_g = 0\}$.

Proposition 5.8. *There is an isomorphism*

$$\begin{aligned} \mathrm{U}(1) \times \mathcal{G}^\perp &\longrightarrow \mathcal{G}_0 \\ (e^{ic}, e^{if}) &\longmapsto e^{i(c+f)} \ . \end{aligned}$$

Proof. Let $h = e^{if}$ and set $\lambda_h = \exp\left(\frac{i}{\mathrm{vol}X}\int_X f d\mathrm{vol}_g\right)$. Note that this is well-defined, since if $e^{if} = e^{if'}$ then $f = f' + 2\pi k$ for $k \in \mathbb{Z}$, hence $\int_X f' d\mathrm{vol}_g = \int_X f d\mathrm{vol}_g + 2\pi k \mathrm{vol}X$. This allows us to write down an inverse of the above map:

$$\begin{aligned} \mathcal{G}_0 &\longrightarrow \mathrm{U}(1) \times \mathcal{G}^\perp \\ h &\longmapsto (\lambda_h, \lambda_h^{-1} h) \ . \end{aligned}$$

Note that

$$\lambda_h^{-1} h = \exp\left(i\left(f - \frac{1}{\mathrm{vol}X}\int_X f d\mathrm{vol}_g\right)\right) =: e^{if'}$$

and

$$\int_X f' d\mathrm{vol}_g = \int_X f - \frac{1}{\mathrm{vol}X} \cdot \mathrm{vol}X \int_X f d\mathrm{vol}_g = 0$$

as required. □

Definition 5.9. A gauge transformation $u \in \mathcal{G}$ is called *harmonic* if $\alpha = u\mathrm{d}u^{-1}$ is harmonic. The set of harmonic gauge transformations will be denoted by $\mathcal{G}^h$.

Lemma 5.10. *$\mathcal{G}^h$ is a subgroup of $\mathcal{G}$.*

Proof. Let $u, v \in \mathcal{G}^h$. Then $(uv)\mathrm{d}((uv)^{-1}) = u\mathrm{d}u^{-1} + v\mathrm{d}v^{-1}$. Since uu^{-1} is constant, $u\mathrm{d}u^{-1} = -u^{-1}\mathrm{d}u$. Thus, $\mathcal{G}^h$ is closed under multiplication and inversion. □

Note that $0 = \mathrm{d}^2(uu^{-1}) = 2(\mathrm{d}u)(\mathrm{d}u^{-1}) = 2\mathrm{d}\alpha$. Hence, for harmonicity one needs to check only $\mathrm{d}^*\alpha = 0$.

Proposition 5.11. *For an arbitrary $u \in \mathcal{G}$, there exists an (up to a constant) unique $f_u : X \to \mathbb{R}$ such that ue^{-if_u} is harmonic.*

In the proof, we will need to use the *Green's operator* for the Laplacian $\Delta : \Omega^k(X) \to \Omega^k(X)$. Let $H : \Omega^k(X) \to \mathcal{H}^k(X)$ be the orthogonal projection to the harmonic subspace.

Theorem 5.12. *There exists a Green's operator G for Δ, given by $G : \Omega^k(X) \to (\mathcal{H}^k(X))^\perp \subset \Omega^k(X)$, which maps α to the unique $\omega \in (\mathcal{H}^k(X))^\perp$ such that $\Delta\omega = \alpha - H(\alpha)$.*

Remark 5.13. The Green's operator satisfies $H + \Delta G = \mathrm{Id} = H + G\Delta$, and $HG = GH$.

Proof of Proposition 5.11. Set $\beta = u\mathrm{d}u^{-1} \in \Omega^1(X)$ and define a function f by $f = iG(\mathrm{d}^*\beta)$.

We know that $H(\mathrm{d}^*\beta) = 0$ because the image of d^* is orthogonal to the harmonic subspace.

Thus, $\Delta f = i(\Delta G + H)(\mathrm{d}^*\beta) = i\mathrm{d}^*\beta$. We claim that $v = ue^{-if}$ is harmonic, i.e. we have a harmonic form

$$v\mathrm{d}v^{-1} = u\mathrm{d}u^{-1} + e^{-if}\mathrm{d}e^{if} = u\mathrm{d}u^{-1} + i\mathrm{d}f = \beta + i\mathrm{d}f \ .$$

As remarked before, all we need to do is to check that $\mathrm{d}^*(v\mathrm{d}v^{-1})$ vanishes. Let's calculate:

$$\mathrm{d}^*(v\mathrm{d}v^{-1}) = \mathrm{d}^*\beta + i\mathrm{d}^*\mathrm{d}f = \mathrm{d}^*\beta - \mathrm{d}^*\beta = 0\ ,$$

so $v\mathrm{d}v^{-1}$ is indeed harmonic. Uniqueness up to constant follows from our definition of f. □

After modifying this function by an appropriate constant, we obtain:

Corollary 5.14. *Given an arbitrary gauge transformation $u : X \to S^1$, there exists a unique $f_u : X \to \mathbb{R}$ such that $\int_X f_u \mathrm{vol}_g = 0$ and ue^{-if} is harmonic.*

Corollary 5.15. *The map*

$$\begin{aligned} \mathcal{G}^\perp \times \mathcal{G}^h &\longrightarrow \mathcal{G} \\ (e^{if}, g) &\longmapsto e^{if} g \end{aligned}$$

is an isomorphism with inverse

$$\begin{aligned} \mathcal{G} &\longrightarrow \mathcal{G}^\perp \times \mathcal{G}^h \\ u &\longmapsto (e^{if_u}, ue^{-if_u})\ . \end{aligned}$$

Recalling that $\mathcal{G}_0 = \mathrm{U}(1) \times \mathcal{G}^\perp$ and $\mathcal{G}/\mathcal{G}_0 \cong H^1(X;\mathbb{Z})$, we conclude $H^1(X;\mathbb{Z}) \cong \mathcal{G}/\mathcal{G}_0 \cong \mathcal{G}^h/\mathrm{U}(1)$. We obtain a short exact sequence

$$1 \longrightarrow \mathrm{U}(1) \longrightarrow \mathcal{G}^h \xrightarrow{\deg} H^1(X;\mathbb{Z}) \longrightarrow 0$$

of Abelian groups ($\mathbb{Z}$-modules). This short exact sequence splits, because there is a section of $\deg$, i.e. a map $v : H^1(X;\mathbb{Z}) \to \mathcal{G}^h$ such that $\deg \circ v = \mathrm{Id}$. Adopting the notation $\mathcal{G}^h_v$ for the image of $H^1(X;\mathbb{Z})$ under v, we then obtain

$$\mathcal{G}^h = \mathcal{G}^h_v \times \mathrm{U}(1) \cong H^1(X;\mathbb{Z}) \times \mathrm{U}(1) = \mathbb{Z}^{b_1(X)} \times \mathrm{U}(1)\ .$$

5.2 The structure of the quotient space $\mathcal{B} \cong \mathcal{C}_\mathfrak{s}/\mathcal{G}$

We now turn our attention to $\mathcal{B} \cong \mathcal{C}_\mathfrak{s}/\mathcal{G}$. Using our discussion of $\mathcal{G}$, we first pass to $\mathcal{C}_\mathfrak{s}/\mathcal{G}^\perp$. This still has a residual action of $\mathcal{G}^h$, and we consider $\mathcal{B}$ as the two-step quotient $(\mathcal{C}_\mathfrak{s}/\mathcal{G}^\perp)/\mathcal{G}^h$.

Theorem 5.16. *The action of $\mathcal{G}^\perp$ on $\mathcal{C}_\mathfrak{s}$ admits a global slice S, i.e. $\mathcal{C}_\mathfrak{s}$ is $\mathcal{G}^\perp$-equivariantly diffeomorphic to $S \times \mathcal{G}^\perp$.*

Proof. Fix a $U(1)$-connection A_0. Let S be the affine space

$$S = \left\{(A_0 + a, \Phi) \in \mathcal{C}_\mathfrak{s}\ \middle|\ a \in i\Omega^1(X) \text{ with } \mathrm{d}^*a = 0\right\}\ .$$

We may consider $\mathrm{d}^*a = 0$ as a "gauge fixing condition"[10], since

$$\begin{aligned}\mathfrak{e} : \mathcal{G}^\perp \times S &\longrightarrow \mathcal{C}_\mathfrak{s} \\ (e^{if}, (A_0 + a, \Phi)) &\longmapsto (A_0 + a - i\mathrm{d}f, e^{if}\Phi)\end{aligned}$$

is a diffeomorphism with inverse

$$\begin{aligned}\mathfrak{e}^{-1} : \mathcal{C}_\mathfrak{s} &\longrightarrow \mathcal{G}^\perp \times S \\ (A_0 + b, \Psi) &\longmapsto (e^{-G(\mathrm{d}^*b)}, (A_0 + b - \mathrm{d}(G(\mathrm{d}^*b)), e^{G(\mathrm{d}^*b)}\Psi)) .\end{aligned}$$

We note that $\mathfrak{e}^{-1}$ is defined in a sensible way: $e^{if} \in \mathcal{G}^\perp$ if $\int_X f \mathrm{vol}_g = 0$, which is equivalent to $H(f) = 0$ since $f \in (\mathcal{H}^0(X))^\perp$ if and only if $\langle f, g\rangle_{L^2} = 0$ for every constant $g \in \mathcal{H}^0(X)$, but constant functions make up $\mathcal{H}^0(X)$. Thus, to see that $\exp(-G(\mathrm{d}^*b)) \in \mathcal{G}^\perp$, we just need $HG(\mathrm{d}^*b)) = 0$, but $HG = GH$ and therefore it suffices that $\mathrm{d}^*b \in (\mathcal{H}^0(X))^\perp$, which we already established.

Similar reasoning shows that $\mathrm{d}^*(b - \mathrm{d}(G(\mathrm{d}^*b))) = \mathrm{d}^*b - (H + \Delta G(\mathrm{d}^*b)) = 0$. Now, one should check that $\mathfrak{e}$ and $\mathfrak{e}^{-1}$ are indeed inverse to each other. This is Exercise 4 on Sheet 8. □

The theorem tells us that $\mathcal{C}_\mathfrak{s}/\mathcal{G}^\perp \cong S$, an affine space. So this is certainly a Banach manifold. Next we need to consider $S/\mathcal{G}^h$. Now the group $\mathcal{G}^h$ does not act freely on all of S, but it does act freely on the subset $S^* := S \cap \mathcal{C}_\mathfrak{s}^*$.

Theorem 5.17. *The quotient space $\mathcal{B}^* := \mathcal{C}_\mathfrak{s}^*/\mathcal{G} = S^*/\mathcal{G}^h$ is a Banach manifold with the weak homotopy type of $\mathbb{C}\mathrm{P}^\infty \times T^{b_1(X)}$, where $T^{b_1(X)}$ is the torus $H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$.*

Proof. The group $\mathcal{G}^h \cong \mathcal{G}^h_v \times \mathrm{U}(1)$ acts freely on S^* and $\mathcal{G}^h_v$ acts properly discontinuously, hence $S^*/\mathcal{G}^h_v$ is Hausdorff. Compactness of the remaining $\mathrm{U}(1)$ then guarantees that $S^*/\mathcal{G}^h$ is Hausdorff as well.

Since there exist explicit local slices for the action $\mathcal{G}^h \curvearrowright S^*$, $\mathcal{B}$ is a Banach manifold. The slice S is an affine space, and therefore contractible. But what about S^*? Its complement $S \setminus S^*$ has infinite codimension in S, and therefore S^* has the same weak homotopy type as S, meaning it is weakly contractible. It follows that $\mathcal{B}^* = S^*/\mathcal{G}^h$ is a classifying space for $\mathcal{G}^h = U(1) \times H^1(X;\mathbb{Z})$. An obvious classifying space for this group is $BS^1 \times B\mathbb{Z}^{b_1(X)} = \mathbb{C}\mathrm{P}^\infty \times T^{b_1(X)}$. The uniqueness of homotopy types of classifying spaces implies that $\mathcal{B}^*$ also has this weak homotopy type. □

Corollary 5.18. *Let $\widetilde{\mathcal{B}^*} = S^*/\mathcal{G}^h_v$. Then the projection $\widetilde{\mathcal{B}^*} \to \mathcal{B}^*$ is a principal S^1-bundle.*

As another consequence of the theorem, note that since the Euler class $e \in H^2(\mathcal{B}^*;\mathbb{Z})$ restricted to $\mathbb{C}\mathrm{P}^\infty$ is a generator of $H^2(\mathcal{B}^*;\mathbb{Z})$, the cohomology ring of $\mathcal{B}^*$ is

$$H^*(\mathcal{B}^*;\mathbb{Z}) = \mathbb{Z}[e] \otimes H^*(T^{b_1(X)};\mathbb{Z}) = \mathbb{Z}[e] \otimes \Lambda^*[a_1, \ldots, a_{b_1(X)}] .$$

[10]The condition $\mathrm{d}^*a = 0$ is sometimes said to specify the Coulomb gauge.

5.3 Reducible solutions and walls in parameter space

Deciding the existence or non-existence of reducible solutions to the Seiberg-Witten equations is a simple application of Hodge theory. We discuss this now, since some of the later discussions of the structure of the moduli space will need to be restricted to (neighbourhoods of) irreducible solutions.

Recall that the reducible solutions are of the form $(A, 0) \in \mathcal{Z}_\omega$. The Dirac equation is then satisfied automatically for any A, and we just need to solve the curvature equation $F^+_{\hat{A}} = \omega$. Since we usually want to *avoid* reducible solutions, we are interested in the following question: Given some (parameter) $\omega \in i\Omega^2_+(X)$, does there exist a connection A with $F^+_{\hat{A}} = \omega$?

We can answer this question using the bilinear pairing

$$\begin{aligned} H^2_{\mathrm{dR}}(X) \times \mathcal{H}^2_+(X) &\longrightarrow \mathbb{R} \\ ([\mu], \tau) &\longmapsto \langle [\mu], \tau \rangle_{L^2} = \int_X \mu \wedge *\tau = \int_X \mu \wedge \tau \,. \end{aligned}$$

Let g be a Riemannian metric on X, and $L \to X$ a complex line bundle.

Definition 5.19. We set $\mathcal{W}_{g,L} = \{\omega \in i\Omega^2_+(X) \mid |\langle \omega + 2\pi i c_1(L), \mathcal{H}^2_+(X) \rangle \equiv 0\}$.

Lemma 5.20. *The subset $\mathcal{W}_{g,L}$ is an infinite-dimensional affine subspace of $i\Omega^2_+(X)$ of codimension $b^+_2(X) = \dim \mathcal{H}^2_+(X)$.*

Proof. The subset $\mathcal{W}_{g,L}$ is defined by $b^+_2(X)$ many linearly independent conditions on ω. □

Theorem 5.21. *For a given complex line bundle $L \to X$ and a Riemannian metric g there exists a $\mathrm{U}(1)$-connection A on L with $F^+_A = \omega$ if and only if $\omega \in \mathcal{W}_{g,L}$.*

The proof of this theorem relies on the following result.

Lemma 5.22. *Let $\beta \in \Omega^2_+(X)$. Then there exists some $\alpha \in \Omega^1(X)$ with $\beta = (\mathrm{d}\alpha)^+$ if and only if $\langle \beta, \mathcal{H}^2_+(X) \rangle \equiv 0$.*

Proof. Using the Hodge decomposition as in proposition 4.6, one sees that $\Omega^2_+(X) = \mathcal{H}^2_+(X) \oplus (\mathrm{d}\Omega^1(X))^+$. □

Proof of Theorem. Let A_0 be any connection on L. Then $A = A_0 + a$ where $a \in i\Omega^1(X)$ has curvature $F_A = F_{A_0} + \mathrm{d}a$. Therefore $F^+_A = F^+_{A_0} + \mathrm{d}^+ a$ and $F^+_A = \omega$ precisely if we can solve the equation $\mathrm{d}^+ a = \omega - F^+_{A_0} = (\omega - F^+_{A_0})^+$.

By the previous lemma, this is possible precisely if $\langle \omega - F_{A_0}, \mathcal{H}^2_+(X) \rangle_{L^2} = 0$ (we added back the anti-self dual part since it does not contribute in any case). From Chern-Weil theory, we know that $[F_{A_0}] = -2\pi i c_1(L)$, hence the ω-perturbed curvature equation has a solution if and only if $\langle \omega + 2\pi i c_1(L), \mathcal{H}^2_+(X) \rangle = 0$, which means that $\omega \in \mathcal{W}_{g,L}$. □

The consequences of the above theorem above can be best understood using the following definition.

Definition 5.23. For a Spin^c-structure $\mathfrak{s}$ on X with characteristic line bundle L, we define the *wall in parameter space* as the set $\mathcal{W}_{\mathfrak{s}} = \{(g, \omega) \in \mathcal{P} \mid \omega \in \mathcal{W}_{g,L}\}$. It is an infinite-dimensional submanifold of $\mathcal{P}$ of codimension $b_2^+(X)$. The connected components of $\mathcal{P} \setminus \mathcal{W}_{\mathfrak{s}}$ are called *chambers*.

Corollary 5.24. *The Seiberg-Witten equations for parameters $(g, \omega) \in \mathcal{P}$ have a reducible solution if and only if $(g, \omega) \in \mathcal{W}_{\mathfrak{s}}$.*

Corollary 5.25. *Depending on the value of $b_2^+(X)$, we have the following cases.*

(i) *If $b_2^+(X) = 0$, then for all $(g, \omega) \in \mathcal{P}$, there are reducible solutions.*

(ii) *In case $b_2^+(X) = 1$, one generically finds oneself outside the wall, but parameters (g_1, ω_1) and (g_2, ω_2) can only be connected by a path avoiding the wall if they are in the same chamber.*

(iii) *For $b_2^+(X) \geq 2$, paths transverse to the wall are actually disjoint from the wall, hence parameters in the complement of the wall can always be connected by a curve avoiding the wall, i.e. there is only one chamber.*

We will see later that this corollary implies that the Seiberg-Witten invariants are independent of parameters if $b_2^+(X) \geq 2$. For $b_2^+(X) = 1$, one has to deal with so-called “wall-crossing” phenomena.

5.4 Kuranishi charts for the moduli space

We now want to see how to obtain finite-dimensional charts for the moduli space. This discussion will lead to the conclusion that, at least in favourable circumstances, the moduli space is (if nonempty) a smooth manifold of a dimension we can calculate explicitly via the index of the linearised Seiberg-Witten equations.

It is at this point that we really need the Banach space setup we have arranged by completing the spaces of smooth objects with respect to suitable Sobolev norms. This makes available the machinery of the implicit function theorem for Fredholm maps between Banach manifolds.

Let $f : X \to Y$ be a smooth map between Banach manifolds, which is Fredholm in the sense that for every $x \in X$ the derivative $\mathcal{T}_x f$ is a Fredholm operator. Fix a point $x_0 \in X$ with image $y_0 = f(x_0) \in Y$ and derivative $L = \mathcal{T}_{x_0} f$. We define $K = \ker \mathcal{T}_{x_0} f$. This has a complement B in the tangent space to X at x_0 so that L restricted to B is an isomorphism onto the image of L. The image in turn has a complement C in the tangent space to Y at y_0, which we can identify with the cokernel of L. The implicit function theorem now says that there exist local charts (U, κ) for X and (V, κ') for Y around x_0 and y_0 respectively such that

$$\begin{aligned} \kappa &: U \to B \oplus K, \text{ mapping } \ x_0 \mapsto 0 \\ \kappa' &: V \to B \oplus C, \text{ mapping } \ y_0 \mapsto 0 \,, \end{aligned}$$

and so that in these charts $F = \kappa' \circ f \circ \kappa^{-1} : B \oplus K \to B \oplus C$ is given on an open set $W \subset B \oplus K$ by $F(b,k) = (L(b), \psi(b,k)) \in B \oplus C$, where $\psi : W \to C$, $(b,k) \mapsto \psi(b,k) =: \psi_b(k)$ is a smooth map. Then we have $f^{-1}(y_0) \cong F^{-1}(0,0) \cong \psi_0^{-1}(0)$. If $C = 0$, then ψ_0 is identically zero, and $f^{-1}(y_0)$ is diffeomorphic to an open neighborhood of 0 in K. It is therefore a smooth manifold of dimension equal to the dimension of K.

For historical reasons, the map $\psi_0 : W' \subset K \to C$ is called a *Kuranishi map*. It is given by the essentially non-linear part of f, that is, the part which is not already encoded in the derivative L.

We now want to apply this discussion to the Seiberg-Witten map f_ω. The first issue is that, because of the gauge-equivariance of f_ω, equivalently the gauge invariance of the Seiberg-Witten equations, the map f_ω is not Fredholm. For every (A, Φ) the tangent space to its gauge orbit is contained in the kernel of the derivative of f_ω given by the linearised equations. So this kernel is always infinite-dimensional. However, if we quotient out the tangent space to the gauge orbit, then the kernel becomes finite-dimensional, and the linearised Seiberg-Witten equations do define a Fredholm operator on $\mathcal{B} \cong \mathcal{C}_\mathfrak{s}/\mathcal{G}$. This was the conclusion of our discussion of the linearised equations in section 4.7, leading to the elliptic complex of proposition 4.40.

Definition 5.26. For $(A, \Phi) \in \mathcal{Z}_\omega$, let $H^i_{(A,\Phi)}$ be the ith cohomology group of the elliptic complex from proposition 4.40.

The index of the complex is $\dim H^0_{(A,\Phi)} - \dim H^1_{(A,\Phi)} + \dim H^2_{(A,\Phi)}$. We have seen already that the index is independent of the choice of (A, Φ), though the individual cohomology groups may well depend on this choice.

Lemma 5.27. *For any configuration (A, Φ) we have*

$$H^0_{(A,\Phi)} \cong \mathfrak{g} = \begin{cases} 0 & \text{if } \Phi \not\equiv 0 \\ \mathbb{R} & \text{if } \Phi \equiv 0 \,. \end{cases}$$

The notation $\mathfrak{g}$ is supposed to remind us that this is the Lie algebra of the stabiliser of (A, Φ) in the gauge group.

Proof. We have $\xi \in H^0_{(A,\Phi)} \Leftrightarrow L_{(A,\Phi)}\xi = (0,0)$. This means that $\mathrm{d}\xi = 0$ and $\xi\Phi = 0$. By the first condition ξ must be (locally, hence globally) constant and if $\Phi \equiv 0$ there is no further condition. If $\Phi \not\equiv 0$, the second equation gives $\xi = 0$. □

Let us assume for the moment that (A, Φ) is an irreducible solution of the SW equations, i.e. $H^0_{(A,\Phi)} = 0$. Let S be a local slice for the $\mathcal{G}$-action on $\mathcal{C}_\mathfrak{s}$ around (A, Φ). This means that a neighborhood of (A, Φ) admits a smoothly embedded (closed) Banach submanifold S and the neighborhood is diffeomorphic to $S \times \mathcal{G}$ (i.e. we can picture S as a transverse submanifold to the $\mathcal{G}$-orbits).

Consider $f_\omega|_S : S \to i\Omega^2_+(X) \times \Gamma(V_-)$. This is a Fredholm map between Banach manifolds, and a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ is equal to $(f_\omega|_S)^{-1}(0)$. The above discussion of the implicit function theorem yields a Kuranishi map $\psi : H^1_{(A,\Phi)} \to H^2_{(A,\Phi)}$ whose zero-set gives a model of a neighbourhood of $[A, \Phi]$ in $\mathcal{M}_\omega$. This makes sense because

$$H^1_{(A,\Phi)} = \ker \mathcal{T}_{(A,\Phi)} f_\omega / \operatorname{im} L_{(A,\Phi)} = \ker \mathcal{T}_{(A,\Phi)}\big(f_\omega|_S\big)$$
$$H^2_{(A,\Phi)} = \operatorname{coker} \mathcal{T}_{(A,\Phi)} f_\omega = \operatorname{coker} \mathcal{T}_{(A,\Phi)}\big(f_\omega|_S\big) \ .$$

Now assume that $H^2_{(A,\Phi)} = 0$ as well—transversality of f_ω to $(0, 0)$ (to be established in theorem 5.30) ensures that this will be true generically. Then an open neighborhood of $[A, \Phi]$ in $\mathcal{M}_\omega$ looks like an open neighborhood of 0 in $H^1_{(A,\Phi)}$; in particular, around $[A, \Phi]$, $\mathcal{M}_\omega$ is a smooth manifold with dimension equal to the dimension of $H^1_{(A,\Phi)}$. Our discussion of the index of the elliptic complex shows:

Proposition 5.28. *If $H^0_{(A,\Phi)} = 0 = H^2_{(A,\Phi)}$, then a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ is a smooth manifold of dimension*

$$\dim^{\exp} \mathcal{M}_\omega := \frac{1}{4}\big(c_1^2(L_\mathfrak{s}) - (2\chi(X) + 3\sigma(X))\big)$$

We call $\dim^{\exp} \mathcal{M}_\omega$ the *expected dimension* of the moduli space.

For reducible solutions, the analogous result is:

Proposition 5.29. *If $H^0_{(A,\Phi)} \cong \mathbb{R}$, and $H^2_{(A,\Phi)} = 0$, then a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ is the quotient of a smooth manifold of dimension $\dim^{\exp} \mathcal{M}_\omega + 1$ by an effective $\mathrm{U}(1)$-action.*

Proof. We still have the Kuranishi map $\psi : H^1_{(A,\Phi)} \to 0$ and the constant gauge transformations (i.e. elements of $\mathcal{G}_{(A,\Phi)}$) act on $\mathcal{A}_\mathfrak{s} \times \Gamma(V_+)$ and $i\Omega^2_+(X) \times \Gamma(V_-)$. The Kuranishi map is $\mathrm{U}(1)$-equivariant because f_ω is, and so its zero-set inherits an $\mathrm{U}(1)$-action. The index of the elliptic complex is now given by

$$\dim H^0_{(A,\Phi)} - \dim H^1_{(A,\Phi)} + \dim H^2_{(A,\Phi)} = 1 - \dim H^1_{(A,\Phi)} \ .$$

Hence

$$\dim H^1_{(A,\Phi)} = 1 + \frac{1}{4}\big(c_1^2(L_\mathfrak{s}) - (2\chi(X) + 3\sigma(X))\big) = 1 + \dim^{\exp} \mathcal{M}_\omega \ .$$

Thus, a neighbourhood of $[A, 0]$ in $\mathcal{M}_\omega$ is diffeomorphic to a $\mathrm{U}(1)$-quotient of an open subset of $H^1_{(A,\Phi)}$, which has dimension $1 + \dim^{\exp} \mathcal{M}_\omega$. □

5.5 Transversality for the Seiberg-Witten map

In this section, we build on the discussion of the previous section where, under the assumption $H^2_{(A,\Phi)} = 0$, we showed that $\mathcal{M}^*_\omega$ locally looks like a manifold of dimension $c_2(V_+)$. In this section, we establish:

Theorem 5.30. *For any fixed Riemannian metric g on X, and a generic $\omega \in i\Omega^2_+(X)$, the irreducible part of the moduli space $\mathcal{M}^*_\omega = (\mathcal{Z}_\omega \cap \mathcal{C}^*_\mathfrak{s})/\mathcal{G}$, is either empty or a smooth manifold of dimension $c_2(V_+)$.*

If we vary the self-dual form that appears in the perturbed curvature equation, we get a collection of moduli spaces. It is useful to collect all this information into a single object, which leads the following definition:

Definition 5.31 (Parametrized moduli space)**.** We define the parametrized moduli space as

$$\mathcal{M} = \left\{([A,\Phi],\omega) \in \mathcal{B} \times i\Omega^2_+(X) \,\middle|\, f_\omega(A,\Phi) = 0\right\} .$$

We also define its irreducible part as

$$\mathcal{M}^* = \mathcal{M} \cap \left(\mathcal{B}^* \times i\Omega^2_+(X)\right) .$$

If $\pi : \mathcal{M}^* \to i\Omega^2_+(X)$ is the canonical projection, observe that $\mathcal{M}^*_\omega = \pi^{-1}(\omega)$. The usefulness of the parametrized moduli space is demonstrated by the following theorem:

Theorem 5.32 (Transversality)**.**

(i) *$\mathcal{M}^*$ is a Banach manifold.*

(ii) *The projection $\pi : \mathcal{M}^* \to i\Omega^2_+(X)$ is Fredholm, with index $c_2(V_+)$.*

(iii) *The set of regular values $\omega \in i\Omega^2_+(X)$ of π is generic. For a regular value ω, $\mathcal{M}^*_\omega = \pi^{-1}(\omega)$ is either empty or a smooth manifold of dimension $c_2(V_+)$.*

Proof.

(i) Pick a local slice S^* for the action $\mathcal{G} \curvearrowright \mathcal{C}^*_\mathfrak{s}$ around $(A_0,\Phi_0) \in \mathcal{Z}_\omega$. Consider the map

$$\begin{aligned} F:\quad S^* \times i\Omega^2_+(X) &\longrightarrow i\Omega^2_+ \times \Gamma(V_-) \\ (A,\Phi,\omega) &\longmapsto (F^+_{\hat{A}} - \sigma(\Phi,\Phi) - \omega, D^+_A\Phi) . \end{aligned}$$

It is clear from lemma 4.38 that its differential is given by

$$\mathcal{T}_{(A,\Phi,\omega)}F(a,\varphi,\tau) = (2\mathrm{d}^+a - \sigma(\Phi,\varphi) - \sigma(\varphi,\Phi) - \tau, D^+_A\varphi + \gamma(a)\Phi) .$$

Our goal is to show that if (A,Φ) solves the ω-perturbed monopole equations (i.e. $F(A,\Phi,\omega) = 0$), then $\mathcal{T}_{(A,\Phi,\omega)}F$ is surjective.

Let $(\eta,\psi) \in i\Omega^2_+(X) \times \Gamma(V_-)$ be orthogonal to the image of $\mathcal{T}_{(A,\Phi,\omega)}F$, i.e.

$$\int_X (\langle 2\mathrm{d}^+a - \sigma(\Phi,\varphi) - \sigma(\varphi,\Phi) - \tau, \eta\rangle + \langle D^+_A\varphi + \gamma(a)\Phi, \psi\rangle)\mathrm{dvol}_g = 0$$

for every (a, φ, τ). Our task is to show that $(\eta, \psi) = (0, 0)$. Note that the second term does not depend on τ, though the first one does. Since we may choose τ at will, η must vanish. We are left with the equation

$$\int_X \langle D_A^+ \varphi, \psi \rangle d\mathrm{vol}_g = - \int_X \langle \gamma(a)\Phi, \psi \rangle d\mathrm{vol}_g .$$

Now note that the right-hand side does not depend on φ, hence

$$\int_X \langle D_A^+ \varphi, \psi \rangle d\mathrm{vol}_g = \int_X \langle \varphi, D_A^- \psi \rangle d\mathrm{vol}_g$$

does not depend on φ. Since we may choose φ arbitrarily, we conclude that $D_A^- \psi = 0$. Finally, we have

$$\int_X \langle \gamma(a)\Phi, \psi \rangle d\mathrm{vol}_g = 0$$

for any $a \in i\Omega^1(X)$. By assumption, $\Phi \not\equiv 0$. Thus, consider a point p such that $\Phi(p) \neq 0$. Choose $a \in \Omega^1(X)$ locally such that $\gamma(a(p))\Phi(p) = \psi(p)$. If $\psi(p) \neq 0$, the integrand will be positive near p, and using a cutoff function we can force the integral to be strictly positive. This is a contradiction, hence $\psi(p) = 0$. Since Φ is nonzero on an open neighborhood U of p, we conclude $\psi|_U \equiv 0$. The unique continuation property for elements in the kernel of an elliptic operator (such as D_A^-) now implies that $\psi \equiv 0$ since $D_A^- \psi = 0$. Thus, $\mathcal{T}_{(A,\Phi,\omega)} F$ is surjective and $(0, 0)$ is a regular value of F. As the preimage of a regular value, $\mathcal{M}^*$ is a Banach manifold.

(ii) We will show that (for $[(A, \Phi)] \in \mathcal{M}^*$—we will omit the square brackets in the rest of this proof) the kernel and cokernel of $\mathcal{T}_{(A,\Phi,\omega)} \pi|_{\mathcal{M}^*}$ (meaning we consider only tangent vectors to $\mathcal{M}^*$) coincide with the kernel and cokernel of $\mathcal{T}_{(A,\Phi)} f_\omega|_S = \mathcal{T}_{(A,\Phi)} f_\omega|_{S^*}$ (since $(A, \Phi) \in \mathcal{C}_{\mathfrak{s}}^*$, $S = S^*$ near (A, Φ)), which in turn are simply $H^1_{(A,\Phi)}$ and $H^2_{(A,\Phi)}$ (cf. lemma 5.27).

Clearly $\mathcal{T}_{(A,\Phi,\omega)} F = \mathcal{T}_{(A,\Phi)} f_\omega|_S - (\tau, 0)$, hence

$$T_{(A,\Phi,\omega)} \mathcal{M}^* = \ker \mathcal{T}_{(A,\Phi,\omega)} F = \{(a, \varphi) \mid \mathcal{T}_{(A,\Phi)} f_\omega(a, \varphi) \in TS^* = (\tau, 0)\}$$

and $\mathcal{T}_{(A,\Phi,\omega)} \pi(a, \varphi, \tau) = \tau$, therefore $(a, \varphi, \tau) \in \ker \mathcal{T}_{(A,\Phi,\omega)} \pi|_{\mathcal{M}^*}$ precisely if $\tau = 0$ and $(a, \varphi) \in \ker T_{(A,\Phi)} f_\omega|_S$ (the latter condition simply expresses that $(a, \varphi, \tau) \in T_{(A,\Phi,\omega)} \mathcal{M}^*$). We conclude that $\ker \mathcal{T}_{(A,\Phi,\omega)} \pi|_{\mathcal{M}^*} \cong \ker \mathcal{T}_{(A,\Phi)} f_\omega|_{S^*} \cong H^1_{(A,\Phi)}$.

Given $(\eta, \psi) \in i\Omega^2_+(X) \times \Gamma(V_-)$, we can use the fact that $(0, 0)$ is a regular value of F to see that we can always find (a, φ, τ) such that $\mathcal{T}_{(A,\Phi)} f_\omega(a, \varphi) = (\eta + \tau, \varphi)$. Therefore, any element of $\operatorname{coker} \mathcal{T}_{(A,\Phi)} f_\omega$ is of the form $(\mu, 0)$. This means that the composite map

$$\begin{array}{cccccc} \kappa : & i\Omega^2_+(X) & \longrightarrow & i\Omega^2_+ \times \Gamma(V_-) & \xrightarrow{\text{proj}} & H^2_{(A,\Phi)} \\ & \mu & \longmapsto & (\mu, 0) & \longmapsto & [(\mu, 0)] \end{array}$$

is surjective. Thus, $H^2_{(A,\Phi)} \cong i\Omega^2_+/\ker\kappa$. The kernel of κ consists of elements represented by $(\mu, 0) \in \operatorname{im} \mathcal{T}_{(A,\Phi)} f_\omega|^*_S$, which is equivalent to the existence of some tuple $(a, \varphi, \mu) \in \mathcal{T}_{(A,\Phi,\omega)}\mathcal{M}^*$. But this precisely means $\mu \in \operatorname{im} \mathcal{T}_{(A,\Phi,\omega)}\pi|_{\mu M^*}$. Thus, $\ker\kappa \cong \operatorname{im} \mathcal{T}_{(A,\Phi,\omega)}\pi|_{\mu M^*}$ and $H^2_{(A,\Phi)} \cong \operatorname{coker} \mathcal{T}_{(A,\Phi,\omega)}\pi|_{\mathcal{M}^*}$.

(iii) This is the Banach manifolds version of Sard's theorem, more precisely, the Sard-Smale theorem for Fredholm maps.

□

We have now proved all the standard results for moduli spaces of solutions to gauge theoretic equations in the case of the Seiberg-Witten equations. The conclusion is that for generic parameters the moduli space of irreducible solutions is either empty, or a smooth manifold whose expected dimension is computed from the index of the linearised equations (modulo gauge). In particular, in cases where the expected dimension is negative, there will be no irreducible solutions at all for generic ω. This is of course in contrast with the case of reducible solutions.

6 Bounds on solutions and compactness of moduli spaces

We now want to prove some important a priori bounds on solutions of the Seiberg-Witten equations, and apply them to see that for fixed parameters the moduli space is always compact. This is very important in applications, and is one of the reasons why the Seiberg-Witten equations are easier to use than, say, the Yang-Mills equations, for which moduli spaces are not usually compact due to Uhlenbeck bubbling.

6.1 A priori bounds

6.1.1 Pointwise bounds

We start by proving some pointwise bounds on solutions. For this we use the maximum principle and the following lemma.

Lemma 6.1. *The Laplacian of $|\Phi|^2$ can be expressed as*

$$\frac{1}{2}\Delta|\Phi|^2 = \operatorname{Re}\langle\nabla_A^*\nabla_A\Phi, \Phi\rangle - |\nabla^A\Phi|^2 \,.$$

Proof. In a local orthonormal frame for TX, and the Laplacian of a smooth function f is given by $\nabla f = -\sum_i \nabla^2_{e_i,e_i} f$, where $\nabla^2_{X,Y} = \nabla_X\nabla_Y f - \nabla_{\nabla_X Y} f = L_X L_Y f - L_{\nabla_X Y} f$. We apply this to $f = |\Phi|^2$:

$$\begin{aligned}\frac{1}{2}\nabla|\Phi|^2 = -\frac{1}{2}\nabla^2_{e_i,e_i}|\Phi|^2 &= -\sum_i \Big(\nabla_{e_i}\operatorname{Re}(\langle\nabla^A_{e_i}\Phi, \Phi\rangle) - \operatorname{Re}(\langle\nabla^A_{\nabla_{e_i}e_i}\Phi, \Phi\rangle)\Big) \\ &= -\sum_i \operatorname{Re}(\langle\nabla^A_{e_i}\nabla^A_{e_i}\Phi, \Phi\rangle + \langle\nabla^A_{e_i}\Phi, \nabla^A_{e_i}\Phi\rangle - \langle\nabla^A_{\nabla_{e_i}e_i}\Phi, \Phi\rangle)\Big) \,.\end{aligned}$$

All that is left to show is that the first and last term combine as follows:

$$-\sum_i \Big(\mathrm{Re}(\langle \nabla^A_{e_i}\nabla^A_{e_i}\Phi,\Phi\rangle - \langle \nabla^A_{\nabla_{e_i}e_i}\Phi,\Phi\rangle)\Big) = \mathrm{Re}\langle \nabla^*_A\nabla_A\Phi,\Phi\rangle \ .$$

But this just the real part of the statement of lemma 4.17. □

This yields an immediate bound on $|\Phi|^2$:

Proposition 6.2. *Consider an L^2_4-solution (A,Φ) to the (unperturbed) monopole equations and let $p \in X$ be a point at which $|\Phi|^2$ attains its maximum. Then $|\Phi(p)|^4 \leq -s_g(p)|\Phi(p)|^2$.*

Note that spinors in L^2_4 are continuous, and therefore $|\Phi|^2$ is a continuous function which attains its maximum by compactness of X.

Proof. At a maximum, $\Delta|\Phi|^2 \geq 0$. Using our lemma, we find (in p):

$$0 \leq \mathrm{Re}\langle \nabla^*_A\nabla_A\Phi,\Phi\rangle - |\nabla^A\Phi|^2 \leq \mathrm{Re}\langle \nabla^*_A\nabla_A\Phi,\Phi\rangle \ .$$

Using the pointwise Weitzenböck formula (theorem 4.15) this becomes

$$0 \leq \mathrm{Re}(\langle D^-_A D^+_A\Phi,\Phi\rangle - \frac{1}{2}\langle\gamma(F^+_{\hat{A}})\Phi,\Phi\rangle) - \frac{1}{4}s_g|\Phi|^2 \ .$$

Using that (A,Φ) is a solution to the SW equations as well as lemma 4.30 we obtain:

$$0 \leq -\frac{1}{4}s_g|\Phi|^2 - \frac{1}{2}\langle(\gamma(\sigma(\Phi,\Phi))\Phi,\Phi) = -\frac{1}{4}s_g|\Phi|^2 - 2|\sigma(\Phi,\Phi)|^2 = -\frac{1}{4}(s_g|\Phi|^2 + |\Phi|^4)$$

whence $|\Phi(p)|^4 \leq -s_g(p)|\Phi(p)|^2$. □

Corollary 6.3. *If $s_g \geq 0$ everywhere on X, then $\Phi \equiv 0$, i.e. every solution to the monopole equations is reducible. If $s_g < 0$ somewhere on X, then*

$$|\Phi|^2 \leq \max_{p\in X}\{-s_g(p)\} > 0 \ .$$

Proof. In the first case, the the right hand of the inequality $|\Phi(p)|^4 \leq -s_g(p)|\Phi(p)|^2$ is non-positive, while the left hand side is non-negative. Therefore both must vanish. So $|\Phi|^2$ vanishes at its maximum, and therefore vanishes identically. The bound $|\Phi|^2 \leq -s_g$ holds at a point p where $|\Phi|^2$ achieves its maximum. This proves the second claim. □

This is a C^0-bound on $|\Phi|^2$. Notice that since $|F^+_{\hat{A}}| = |\sigma(\Phi,\Phi)| = |\Phi|^2/2\sqrt{2}$ (by the curvature equation and lemma 4.30), we automatically obtain a C^0-bound for $|F^+_{\hat{A}}|$.

Next, we look for analogous results for the perturbed SW equations. Let $p \in X$ be a maximum of $|\Phi|^2$ (with (A,Φ) a solution); going through the same steps, we obtain an extra term:

$$0 \leq \frac{1}{2}\Delta|\Phi|^2 \leq -\frac{1}{4}(s_g|\Phi|^2 + |\Phi|^4) - \frac{1}{2}\langle\gamma(\omega)\Phi,\Phi\rangle \ .$$

Using lemma 4.30 and the Cauchy-Schwarz inequality, we find

$$\begin{aligned}
0 &\le -\frac{1}{4}(s_g|\Phi|^2 + |\Phi|^4) - 2\langle\omega, \sigma(\Phi,\Phi)\rangle \\
&\le -\frac{1}{4}(s_g|\Phi|^2 + |\Phi|^4) + 2|\omega||\sigma(\Phi,\Phi)| \\
&= -\frac{|\Phi|^2}{4}(s_g + |\Phi|^2 - 2\sqrt{2}|\omega|) \ .
\end{aligned}$$

We find that at the point p we have $|\Phi|^4 \le -|\Phi|^2(s_g - 2\sqrt{2}|\omega|)$. This shows:

Proposition 6.4. *Consider the function $s_{g,\omega} = \min_{p\in X}\{0, s_g(p) - 2\sqrt{2}|\omega(p)|\}$ depending only on $(g,\omega) \in \mathcal{P}$. Let $(A,\Phi) \in \mathcal{Z}_\omega$, i.e. (A,Φ) is a solution of the Seiberg-Witten equations for the parameters (g,ω). Then (A,Φ) satisfies the pointwise bounds*

$$\begin{aligned}
|\Phi|^2 &\le -s_{g,\omega} \\
|F^+_{\hat{A}}| &\le -\frac{1}{2\sqrt{2}}s_{g,\omega} + \max_{p\in X}|\omega| \ ,
\end{aligned}$$

where we used that $\omega \in L^2_3$, which embeds in C^0, to see that it assumes its maximum.

6.1.2 Integral bounds

We now turn to integral rather than pointwise bounds on solutions.

Proposition 6.5. *Let (A,Φ) be a solution to the monopole equations for parameters $(g,\omega) \in \mathcal{P}$. Then*

$$\begin{aligned}
\|\Phi\|^2_{L^4} &\le \left\| -s_g + 2\sqrt{2}|\omega| \right\|_{L^2} \\
\left\|\nabla^A\Phi\right\|_{L^2} &\le \frac{1}{2}\left\| -s_g + 2\sqrt{2}|\omega| \right\|_{L^2} \\
\left\|F^+_{\hat{A}}\right\|_{L^2} &\le \frac{1}{2\sqrt{2}}\left\| -s_g + 2\sqrt{2}|\omega| \right\|_{L^2} + \|\omega\|_{L^2} \ .
\end{aligned}$$

Proof. We consider the integrated Weitzenböck formula and use the usual spinor identities:

$$\begin{aligned}
0 &= \int_X \langle D^-_A D^+_A\Phi, \Phi\rangle d\mathrm{vol}_g = \int_X (|\nabla^A\Phi|^2 + \frac{1}{4}s_g|\Phi|^2 + \frac{1}{2}\langle\gamma(F^+_{\hat{A}})\Phi,\Phi\rangle)d\mathrm{vol}_g \\
&= \int_X (|\nabla^A\Phi|^2 + \frac{1}{4}s_g|\Phi|^2 + \frac{1}{4}|\Phi|^4 + 2\langle\omega, \sigma(\Phi,\Phi)\rangle)d\mathrm{vol}_g \ .
\end{aligned}$$

Discarding the first term and using the Cauchy-Schwarz inequality twice (in different incarnations), we have:

$$\begin{aligned}\int_X |\Phi|^4 d\mathrm{vol}_g &\leq -\int_X (s_g|\Phi|^2 + 8\langle\omega, \sigma(\Phi,\Phi)\rangle) d\mathrm{vol}_g \\ &\leq \int_X (-s_g|\Phi|^2 + 8|\omega||\sigma(\Phi,\Phi)|) d\mathrm{vol}_g = \int_X |\Phi^2|(-s_g + 2\sqrt{2}|\omega|) d\mathrm{vol}_g \\ &\leq \left(\int_X |\Phi|^4 d\mathrm{vol}_g\right)^{1/2} \left(\int_X (-s_g + 2\sqrt{2}|\omega|)^2 d\mathrm{vol}_g\right)^{1/2} .\end{aligned}$$

Thus, we obtain

$$\|\Phi\|_{L^4}^2 \leq \| - s_g + 2\sqrt{2}|\omega|\|_{L^2} ,$$

which was the first claim. For the second inequality, discard the $|\Phi|^4$-term instead of the $\nabla^A\Phi$-term and proceed analogously. For the last bound, we once again need Cauchy-Schwarz:

$$\begin{aligned}\int_X |F_{\hat{A}}^+|^2 d\mathrm{vol}_g &= \int_X |\sigma(\Phi,\Phi) + \omega|^2 d\mathrm{vol}_g \\ &= \int_X \Big(\frac{1}{8}|\Phi|^4 + 2\langle\sigma(\Phi,\Phi),\omega\rangle + |\omega|^2\Big) d\mathrm{vol}_g \\ &\leq \int_X \Big(\frac{1}{8}|\Phi|^4 + \frac{1}{\sqrt{2}}|\Phi|^2|\omega| + |\omega|^2\Big) d\mathrm{vol}_g \\ &= \int_X \Big(\frac{1}{2\sqrt{2}}|\Phi|^2 + |\omega|\Big)^2 d\mathrm{vol}_g \\ &= \Big\|\frac{1}{\sqrt{8}}|\Phi|^2 + |\omega|\Big\|_{L^2}^2 .\end{aligned}$$

Taking the square root and using the triangle inequality as well as our first bound, we obtain the required result:

$$\|F_{\hat{A}}^+\|_{L^2} \leq \frac{1}{2\sqrt{2}}\| - s_g + 2\sqrt{2}|\omega|\|_{L^2} + \|\omega\|_{L^2} .$$

□

6.2 Easy applications of the bounds

The following statement was Exercise 1 on Sheet 7:

Lemma 6.6. *If a closed oriented 4-manifold is endowed with a metric of non-negative scalar curvature, then all solutions to the unperturbed Seiberg-Witten equations are reducible.*

This is of course an immediate consequence of the C^0 bound we proved above. When you did this exercise, we were not expecting you to do this argument with the maximum principle, but just to take the Weitzenböck formula and integrate it as at the beginning of the proof of Proposition 6.5. This immediately proves that the spinor in any solution vanishes identically.

This lemma is crucial for many applications of Seiberg-Witten theory to the Riemannian geometry of 4-manifolds. Our next statement is a general one, without any Riemannian assumption, which will ultimately give a finiteness statement for Seiberg-Witten invariants.

Corollary 6.7. *Let X be a closed, oriented 4-manifold. For fixed parameters $(g, \omega) \in \mathcal{P}$, there exist at most finitely many* Spin^c*-structures on X such that $\mathcal{M}_\omega \neq \varnothing$ and $\dim^{\exp}\mathcal{M}_\omega \geq 0$.*

Proof. Recall that $\dim^{\exp}\mathcal{M}_\omega \geq 0$ is equivalent to $c_1^2(L_\mathfrak{s}) \geq 2\chi(X) + 3\sigma(X)$. We saw in the proof of proposition 4.29 that Chern-Weil theory implies $4\pi^2 c_1^2(L_\mathfrak{s}) = \|F_{\hat{A}}^+\|_{L^2}^2 - \|F_{\hat{A}}^-\|_{L^2}^2$ and therefore our L^2-bound on $F_{\hat{A}}^+$ implies an L^2-bound from above on $F_{\hat{A}}^-$, depending only on (g, ω).

Now we see that $c_1(L_\mathfrak{s})_\mathbb{R} \in H^2(X; \mathbb{R})$ is contained in a bounded box, because its projections to the spaces of SD and ADS harmonic forms are bounded. Since $c_1(L_\mathfrak{s}) \in H^2(X; \mathbb{Z})$ and the free part of $H^2(X; \mathbb{Z}) \subset H^2(X; \mathbb{R})$ is a lattice while the torsion is finite, only finitely many different first Chern classes are possible. This means that there are only finitely many Spin^c-structures satisfying the assumptions. □

Corollary 6.8. *Assume $b_2^+(X) > 0$, and $(g, \omega) \in \mathcal{P}$ is generic. Then there are at most finitely many* Spin^c*-structures with $\mathcal{M}_\omega \neq \varnothing$.*

Proof. Under the given assumptions, there are no reducible solutions, and solutions are transverse zeros of f_ω. Hence $\mathcal{M}_\omega$ is a smooth manifold and $\dim^{\exp}\mathcal{M}_\omega \geq 0$. The conclusion now follows from the previous corollary. □

6.3 Compactness

When talking about compactness of the moduli space, we will always mean *sequential* compactness, i.e. any sequence has a convergent subsequence. The proof of this requires the following standard result from analysis.

Theorem 6.9 (Sobolev Embedding)**.** *Let X be a compact n-manifold. Then*

- *There exists an embedding $L_{j+m}^p(X) \subset C^j(X)$ if $mp \geq n$.*
- *This embedding is compact if $mp > n$.*

Corollary 6.10. *Let X be a compact 4-manifold. Then there exists a compact embedding $L_k^2(X) \subset C^{k-3}(X)$. Hence, every bounded sequence in $L_k^2(X)$ has a convergent subsequence in C^{k-3}.*

Proof. Here, $n = 4$, $p = 2$ and $m = 3$. □

The precise statement that we will prove is:

Theorem 6.11. *Let (A_i, Φ_i) be a sequence of L^2_4 solutions to the Seiberg-Witten equations. Then there exists a sequence u_i of L^2_5 gauge transformations such that $u_i(A_i, \Phi_i)$ is a bounded sequence in L^2_k for all k. Hence the solutions $u_i(A_i, \Phi_i)$ are C^∞ and there is a subsequence that converges in the C^∞ topology to a C^∞ solution (A, Φ) of the monopole equations. In particular, the moduli space is sequentially compact in the C^∞ topology.*

To establish compactness we need bounds on $\|A\|_{L^2_k}$, and $\|\Phi\|_{L^2_k}$ (up to gauge transformation) that depend only on (X, g) and k. To begin with note that all of the bounds obtained above depend only on (X, g) in the case of the unperturbed SW equations; we now restrict to this case. Given that we proved the corresponding bounds also for the perturbed equations, the proof of compactness also works in that case; only the notation is more complicated in the general case.

We will need to use three following results, in which we use c, c' to denote several (different) constants whose values are of no importance.

Theorem 6.12 (Elliptic Estimate)**.** *Since the Dirac operator is elliptic, it satisfies*

$$\|\Phi\|_{L^p_{k+1}} \leq c \left(\|D^+_{A_0}\Phi\|_{L^p_k} + \|\Phi\|_{L^p} \right) .$$

The symbol of the Dirac operator is Clifford multiplication, and this is an isomorphism, showing that D is elliptic.

Theorem 6.13. *Let P denote the L^2-projection onto the kernel of a linear elliptic first order differential operator $\mathfrak{L}$. Then*

$$\|\Phi - P\Phi\|_{L^p_{k+1}} \leq c\|\mathfrak{L}\Phi\|_{L^p_k} .$$

For our purposes, the cases $\mathfrak{L} = D_{A_0}$ and $\mathfrak{L} = \mathrm{d}^+ \oplus \mathrm{d}^*$ (rolled up half de Rham complex) are relevant.

Theorem 6.14 (Gauge Fixing)**.** *Let $E \to X$ be a complex line bundle and $\hat{A}_0$ a fixed, smooth $\mathrm{U}(1)$-connection. Up to L^2_{k+2} gauge transformations we can write an arbitrary L^2_{k+1} connection $\hat{A}$ as $\hat{A} = \hat{A}_0 + a$, with $\mathrm{d}^* a = 0$ and*

$$\|a\|^2_{L^2_{k+1}} \leq c \left\| F^+_{\hat{A}} \right\|^2_{L^2_k} + c'$$

This follows from our discussion of the Coulomb gauge in the proof of Theorem 5.16.

After these preliminary remarks we now start the proof of compactness. The first monopole equation can be rewritten as

$$D^+_{A_0}\Phi = -\gamma(a)\Phi$$

and the above theorems then allow us to deduce the following important result:

Corollary 6.15 (Bootstrapping)**.** *Suppose a, Φ are bounded in L^2_3 by constants c, c'. Then they are bounded by $c(k)$, $c'(k)$ in L^2_k for all $k \geq 3$.*

Proof. The Sobolev multiplication theorem implies that the multiplication $L^2_k \times L^2_k \to L^2_k$ for every $k \geq 3$ is bounded. Assuming L^2_3-bounds on a, Φ, we obtain bounds on $\gamma(a)\Phi$ and $\sigma(\Phi, \Phi)$ and therefore on $D^+_{A_0}\Phi$ and $F^+_{\hat{A}}$. Using the elliptic estimate, Φ is then also bounded in L^2_4. By gauge fixing, a is bounded in L^2_4 as well. Now we can go through the same steps and inductively obtain bounds for any $k \geq 3$. □

All that is left to establish compactness of the moduli space is:

Theorem 6.16. *There exist constants c, c' depending only on (X, g) such that any solution of the SW equations is gauge equivalent to a solution (A, Φ) with $A = A_0 + a$, $\mathrm{d}^*a = 0$ and*

$$\|a\|_{L^2_3} \leq c$$
$$\|\Phi\|_{L^2_3} \leq c' \ .$$

To prove this, we start with the observation that our L^2-bound on $F^+_{\hat{A}}$ yields an L^2_1-bound on a. To get an L^2_2-bound, we use:

Proposition 6.17. *There exists a constant c depending only on (X, g) such that for any $(A, \Phi) \in \mathcal{Z}_\omega$ we have $\|F_{\hat{A}}\|^2_{L^2_1} \leq c$.*

Proof. Using $F^+_{\hat{A}} = \sigma(\Phi, \Phi)$, combined with the fact that we have both an L^∞-bound on Φ and an L^2-bound on $\nabla^A\Phi$, we get an L^2-bound on $\nabla F^+_{\hat{A}}$, where ∇ is induced by the Levi-Civita connection. Thus, we have an L^2-bound on $\mathrm{d}F^+_{\hat{A}}$, which is the same as an L^2_1-bound on $F^+_{\hat{A}}$. □

We have now established:

Corollary 6.18. *There exists a constant c depending only on (X, g) such that any $(A, \Phi) \in \mathcal{Z}_\omega$ is gauge equivalent to a solution (A, Φ) with $A = A_0 + a$, $\mathrm{d}^*a = 0$, and $\|a\|^2_{L^2_2} \leq c$.*

Proof of Theorem 6.11. There is an L^2_1-bound on Φ since we have L^2-bounds on Φ, a and $\nabla^A\Phi$. Now we use Sobolev multiplication to see that $L^2_2 \times L^\infty \to L^4$, $(a, \Phi) \mapsto -\gamma(a)\Phi = D^+_{A_0}\Phi$ is bounded. The orthogonal projection theorem for $D^+_{A_0}$ then yields an L^4_1-bound on Φ.

Now, we use the same steps to get different L^p_k bounds. Every time, we need a Sobolev multiplication theorem. First, we have:

$$\begin{aligned} L^2_2 \times L^4_1 &\longrightarrow L^3_1 \\ (a, \Phi) &\longmapsto -\gamma(a)\Phi \end{aligned}$$

which yields an L^3_2-bound on Φ. Then

$$\begin{aligned} L^2_2 \times L^3_2 &\longrightarrow L^2_2 \\ (a,\Phi) &\longmapsto -\gamma(a)\Phi \end{aligned}$$

gives an L^2_3-bound. Finally, we find an L^2_3-bound on $F^+_{\hat{A}}$ by using the map

$$\begin{aligned} L^2_3 \times L^2_3 &\longrightarrow L^2_3 \\ (\Phi,\Phi) &\longmapsto \sigma(\Phi,\Phi) = F^+_{\hat{A}} \end{aligned}$$

This establishes the L^2_3-bound on a (which is an element of $L^2_4(i\Omega^1(X))$). □

7 Proof of Donaldson's Theorem

Recall that Donaldson's Theorem 3.16 says that if a smooth closed oriented 4-manifold X has a definite intersection form, then the form is equivalent over $\mathbb{Z}$ to the diagonal form (of the appropriate rank). The original proof used the 1-instanton moduli space for the Yang-Mills equations. We now give a proof using the Seiberg-Witten equations.

Without loss of generality, we may assume $b_2^+(X) = 0$ since we may choose the orientation to make Q_X negative definite rather than positive definite. Our method of proof will be to add simplifying assumptions along the way until we arrive at a contradiction, and then backtrack to remove (or justify!) the assumptions one by one.

Assumption 1: $b_1(X) = 0$.

Fix a Riemannian metric g and a Spin^c-structure $\mathfrak{s}$; the latter exists by theorem 3.40. We consider now the SW equations for $\mathfrak{s}$ and parameters $(g,\omega) \in \mathcal{P}$. Since $b_2^+(X) = 0$, there always exists a reducible solution $(A, 0) \in \mathcal{Z}_\omega$. Finding this solution amounts to solving the curvature equation $F^+_{\hat{A}} = \omega$. From Chern-Weil theory, we know that $[\frac{i}{2\pi}F_{\hat{A}}] = c_1(L_\mathfrak{s})$. Every closed 2-form representing this cohomology class is the curvature of a suitable A, since if A_0 is some fixed Spin^c-connection, the (self-dual) curvature of $\widehat{A_0 + a}$ is $F^+_{\hat{A}} = F^+_{\hat{A}_0} + 2\mathrm{d}^+ a$ so we must solve

$$2\mathrm{d}^+ a = \omega - F^+_{\hat{A}_0} \ .$$

Since $b_2^+(X) = 0$, $\mathrm{d}^+ : i\Omega^1 \to i\Omega^2_+$ is surjective (cf. section 4.1). Thus, we obtain a reducible solution and $\mathcal{M}_\omega \neq \varnothing$. Moreover, the reducible solution is unique up to gauge equivalence since if a, a' are solutions to the curvature equation, then $a - a' \in i\Omega^1(X)$ is d^+-closed, hence d-closed, and therefore exact because $b_1(X) = 0$ by assumption. However, exact forms can be gauged away. Recall that

$$\dim^{\exp}\mathcal{M}_\omega = \frac{1}{4}\left(c_1^2(L_\mathfrak{s}) - 2(\chi(X) + 3\sigma(X))\right) \ .$$

For every Spin^c-structure we know that $c_1(L_\mathfrak{s})$ is a lift of $w_2(X)$ to integral coefficients. This implies that $c_1^2(L_\mathfrak{s}) = \sigma(X) + 8k$ for some $k \in \mathbb{Z}$. Using $\sigma(X) = -b_2^-(X) = -b_2(X)$, we find:

$$\dim \mathcal{M}_\omega = \frac{1}{4}(\sigma(X) + 8k - 2(2 - 2b_1(X) + b_2^+(X) + b_2^-(X)) - 3\sigma(X)) \, , = \frac{1}{4}(8k-4) = 2k - 1$$

which is odd. Now, we add the next assumption:

Assumption 2: $k > 0$, equivalently, $\dim \mathcal{M}_\omega > 0$.

By transversality (see theorem 5.30), we may assume that $\mathcal{M}^*_\omega$ is smooth and of the expected dimension, since the reducible point (which sits inside a space of *positive* dimension) must be a deformation of irreducible solutions. Moreover $\mathcal{M}_\omega$ is compact with one singular point, corresponding to the unique up to gauge reducible solution $(A, 0)$.

We need to understand what a neighborhood of the singular point looks like. At $(A, 0)$, the linearised equations (cf. 4.40) decouple (crucially using $\Phi \equiv 0$) into

$$i\Omega^0 \cong \mathbb{R} \xrightarrow{-\mathrm{d}} i\Omega^1(X) = 0 \xrightarrow{\mathrm{d}^+} i\Omega^2_+ = 0$$

$$0 \xrightarrow{\qquad\quad} \Gamma(V_+) \xrightarrow{D_A^+} \Gamma(V_-) \, .$$

Recall from the Atiyah-Singer index theorem that

$$\mathrm{ind}_\mathbb{C}\, D_A^+ = \frac{1}{8}(c_1^2(L_\mathfrak{s}) - \sigma(X)) = k > 0 \, .$$

We make our final assumption.

Assumption 3: D_A^+ is surjective.

In this case, $\mathrm{ind}_\mathbb{C}\, D_A^+ = \dim_\mathbb{C} \ker D_A^+$, i.e. $\ker D_A^+ \cong \mathbb{C}^k$. At the reducible point, the stabilizer subgroup of $\mathcal{G}$ is $\mathcal{G}_{(A,0)} = \mathrm{U}(1)$, given by the constant gauge transformations that rotate Φ (which now vanishes). Following the discussion after lemma 5.27, a neighborhood of $[A, \Phi] \in \mathcal{M}_\omega$ is given by $\psi^{-1}(0)/S^1$, where $\psi : H^1_{(A,0)} = \ker D_A^+ \to H^2_{(A,0)}$ is a choice of Kuranishi map. Our description of the stabilizer subgroup shows that the differential of ψ may fail to be surjective at $0 \in H^1_{(A,0)}$, but on $H^1_{(A,0)} \setminus \{0\}$, ψ is transverse to $0 \in H^2_{(A,0)}$. When assumption 3 is satisfied, a neighborhood of $[A, 0]$ in $\mathcal{M}_\omega$ is therefore a cone on $\mathbb{C}\mathrm{P}^{k-1}$. The cone point corresponds to the reducible solution.

We can cut out a neighborhood of the cone point, i.e. "truncate" $\mathcal{M}_\omega$ by removing an open cone around the singular point, we get a compact manifold with boundary $\mathbb{C}\mathrm{P}^{k-1}$. If k is *odd*, and this manifold is *orientable*, we obtain a contradiction since the truncated moduli space would demonstrate that $\mathbb{C}\mathrm{P}^{k-1}$ is null-cobordant, which is false since $\sigma(\mathbb{C}\mathrm{P}^{k-1}) \neq 0$ and the signature is a cobordism invariant.

Now, we begin the process of reconsidering our assumptions. If $\mathcal{M}^*_\omega$ is orientable, we also get a contradiction for k even, as we show now. Instead of looking at solutions modulo $\mathcal{G}$, we can look at solutions modulo $\mathcal{G}^\perp = \{u \in \mathcal{G} \mid u = e^{if} \text{ with } \int_X f \mathrm{vol}_g = 0\}$. We denote the moduli space of solutions modulo $\mathcal{G}^\perp$ by $\overline{\mathcal{M}}$.

Recall that $\mathcal{G} \cong \mathcal{G}^\perp \times \mathcal{G}^h$ and that $b_1(X) = 0$, hence $\mathcal{G} = \mathcal{G}_0$, i.e. every $u : X \to S^1$ is of the form $u = e^{if}$ for $f : X \to \mathbb{R}$. The condition $u \in \mathcal{G}^h$ means that $u\mathrm{d}u^{-1} = -i\mathrm{d}f$ is harmonic. But then $\langle \mathrm{dd}^*\mathrm{d}f, \mathrm{d}f\rangle = 0$, which implies that $\mathrm{d}^*\mathrm{d}f = 0$. Repeating this argument shows that $\mathrm{d}f = 0$, i.e. f is constant. This shows that $\mathcal{G}^h \cong U(1)$ in our situation.

It is now clear that away from the singular point $[A, 0]$, we have a circle bundle $\beta : \overline{\mathcal{M}}^*_\omega \to \mathcal{M}^*_\omega$. This moduli space $\overline{\mathcal{M}}_\omega$ is a closed, oriented manifold and therefore has even Euler characteristic. On the other hand, as a space admitting a $U(1)$-action with a single fixed point, it has Euler characteristic 1, a contradiction. Regarding orientability of $\mathcal{M}^*_\omega$, it is a fact (which we will not prove) that $\mathcal{M}^*_\omega$ *is* actually orientable, so this "assumption" need not be relaxed.

Consider assumption 3: If it is not satisfied, we have a Kuranishi map

$$\psi : \mathbb{C}^{k+r} = H^1_{(A,0)} \to H^2_{(A,0)} = \mathbb{C}^r \ .$$

It is still S^1-equivariant, with $\psi(0) = 0$ and $\psi \pitchfork 0 \in \mathbb{C}^r$ on $\mathbb{C}^{k+r} \setminus \{0\}$. A neighborhood of $[A, 0]$ in $\mathcal{M}_\omega$ is given by $\psi^{-1}(0)/S^1$. Moreover, ψ descends to a section of the vector bundle given by $H^2_{(A,0)}$ over $H^1_{(A,0)}/S^1 = C(\mathbb{C}\mathrm{P}^{k+r-1})$ ($C(X)$ denotes the cone over X). If $r = 1$, this would be the hyperplane bundle H but the constant gauge transformations making up S^1 act like p copies of the action on $\mathbb{C}$, i.e. we have a rank r bundle

$$H \oplus \cdots \oplus H \to C(\mathbb{C}\mathrm{P}^{k+r-1}) \ .$$

In the homology of $\mathbb{C}\mathrm{P}^{k+r-1}$, $\psi^{-1}(0)$ is Poincaré dual to x^r, where x is a generator of $H^2(\mathbb{C}\mathrm{P}^{k+r-1})$. For a generic value of the cone parameter, $\psi^{-1}(0)$ is a smooth submanifold N with homology class of a linear $\mathbb{C}\mathrm{P}^{k-1} \subset \mathbb{C}\mathrm{P}^{k+r-1}$. This implies

$$\langle e(\beta)^{k-1}, [N]\rangle = \langle e(\beta)^{k-1}, [\mathbb{C}\mathrm{P}^{k-1}]\rangle \neq 0$$

and we obtain a contradiction because β extends over the compact manifold obtained by truncation, so we should have

$$\langle e(\beta)^{k-1}, [\mathbb{C}\mathrm{P}^{k-1}]\rangle = \langle e(\beta)^{k-1}, \partial(\mathcal{M}^{\mathrm{trunc}}_\omega)\rangle = 0 \ .$$

This is really the same argument as before, instead now we are not arguing that the manifold N obtained as the boundary of the truncated moduli space is null-cobordant, but instead we argue that the circle bundle over it is not null-cobordant as a bundle because its Euler class is non-zero.

Assumption 1 is actually not an assumption because of the following lemma.

Lemma 7.1. *If Q_X is the intersection form of a smooth manifold X, then it is (isomorphic over $\mathbb{Z}$ to) the intersection form of a smooth manifold Y with $b_1(Y) = k$ for any $k \in \mathbb{N}_0$.*

Proof. Raising the first Betti number without changing the intersection form is easy: $b_1(X\#k(S^1 \times S^3)) = b_1(X) + k$ and $S^1 \times S^3$ has no second cohomology, hence $Q_{X\#k(S^1\times S^3)} \cong Q_X$.

Lowering the first Betti number is done by means of surgery. Suppose $b_1(X) > 0$ and $H_1(X) \cong \mathbb{Z}^{b_1(X)} \oplus \mathrm{Tor}$. Pick a basis element $\alpha \in H^1(X)$ for the free part: It can be realized by a smoothly embedded $S^1 \hookrightarrow X$. Pick a tubular neighborhood $T \cong S^1 \times D^3$ of the S^1 in X. Then

$$\partial T = \partial(S^1 \times D^3) = S^1 \times S^2 = \partial(D^2 \times S^2)$$

so we perform surgery. Remove T from X and glue in $B^2 \times S^2$ in its place. In the manifold Y obtained in this way, the circle we started with is null-homotopic, and therefore null-homologous. This shows that $b_1(Y) = b_1(X) - 1$. A Mayer-Vietoris argument shows that $H_2(X) \cong H_2(Y)$ and $Q(Y) \cong_{\mathbb{Z}} Q(X)$. □

Remark 7.2. Note that this Lemma does not contradict Example 3.30, where we saw that the definiteness of the intersection form does exclude certain fundamental groups from appearing.

Finally, we discuss Assumption 2, that $k > 0$. Here we have the following:

Proposition 7.3 (Elkies)**.** *Let Q be a negative definite unimodular symmetric bilinear form over $\mathbb{Z}$. There exists a v with $v \cdot x \equiv x^2 \mod 2$ for all x and $v^2 > -\operatorname{rank} Q$ if and only if Q is not diagonalisable.*

Recall from proposition 3.37 that $w_2(X) \cdot \alpha = r(\alpha \cdot \alpha)$ (r denotes reduction modulo 2) for every $\alpha \in H^2(X;\mathbb{Z})$. Thus, if (and only if) Q_X is not diagonal over $\mathbb{Z}$, there exists some $c \in H^2(X;\mathbb{Z})$ with $r(c) = w_2(X) \in H^2(X;\mathbb{Z}_2)$ and $c^2 + b_2(X) = 8k > 0$. This c determines a Spin^c structure $\mathfrak{s}$ with characteristic line bundle defined by $c_1(L_{\mathfrak{s}}) = c$.

Let us summarise the logic of the proof. If we assume the intersection form of X is negative definite but not diagonalizable, then by Elkies's proposition we find a Spin^c structure $\mathfrak{s}$ on X for which the moduli space is non-empty and of positive dimension, and, by the above argument, leads to a contradiction. Thus, the assumption that the intersection form of X is not diagonalisable was false. This contradiction proves Donaldson's theorem.

7.1 Comparison with Donaldson's argument

The original proof, due to Donaldson, precedes Seiberg-Witten theory and is much more natural. It uses *instantons*. Let X be connected compact smooth oriented 4-manifold with $b_1(X) = 0$ and $b_2^+(X) = 0$. Let P be an $SU(2)$-principal bundle over X with $c_2(P) = 1$ and consider the anti self-duality equation $*F_A = -F_A$; its moduli space of solutions $\mathcal{M}_1$ is the 1-instanton moduli space $\mathcal{M}_1$. It turns out $\dim \mathcal{M}_1 = 5$ and that it *fails* to be compact. Understanding the non-compactness is key.

It turns out that the moduli space only has one "end", which looks like $X \times [0, 1)$ since the non-compactness arises only from concentrating arbitrarily much curvature at a single point. Such configurations are parametrized by a center in X and a scale in $[0, 1)$. (This comes from Uhlenbeck compactness, which is really a failure of compactness, through bubbling of curvature.)

The bundle P splits as $P = L \oplus L^{-1}$ (i.e. $c_1(P) = 0$) and $c_2(P) = 1$ implies that $c_1^2(L) = -1$. Each such line bundle over X gives rise to a cone over $\mathbb{C}\mathrm{P}^2$ with a singular cone point, i.e. the number of singular points in $\mathcal{M}_1$ is equal to the number of classes (up to sign) $\pm c \in H^2(X;\mathbb{Z})$ such that $c^2 = -1$. Cutting away neighborhoods of the singular points, the truncated moduli space becomes an oriented cobordism between X and a disjoint union of p copies of $\mathbb{C}\mathrm{P}^2$'s and q copies of $\overline{\mathbb{C}\mathrm{P}^2}$'s.

Since the complement of the classes that give rise to the cones is of non-negative dimension, the number of singular points is less or equal to $\mathrm{rank}\, H^2(X;\mathbb{Z})$, i.e. $p + q \leq b_2(X)$. At the same time, cobordism invariance of the signature implies $p - q = \sigma(X)$. Now take Q_X to be negative-definite. Then $q - p = b_2(X) \geq p + q$ and therefore $2p \leq 0$, hence $p = 0$. Now, one needs to check that the classes $\pm c$ are mutually orthogonal to conclude that Q_X is diagonal.

Donaldson's argument is very natural, exploiting the non-compactness of the 1-instanton moduli space and the cobordism-invariance of the signature. By contrast, the Seiberg-Witten argument is much more contrived. It is a proof by contradiction, leading to the conclusion that the manifold X with non-diagonal intersection form does not exist, and therefore the moduli space with all its complicated structure that produced the contradiction does not exist either. Elkies's Proposition is pure algebra, in the spirit of the Hasse-Minkowski classification, but was not known before the advent of Seiberg-Witten theory. It was only when geometers tried to prove Donaldson's theorem via Seiberg-Witten theory that the statement of that Proposition came into focus, and Elkies proved it in response to a question from Tom Mrowka.

8 Seiberg-Witten invariants

In this chapter we define the Seiberg-Witten invariants of a closed connected smooth oriented 4-manifold X, study their most basic properties, and give some applications. To define the invariants, we would like to have smooth moduli spaces of irreducible solutions for generic parameters, and this forces us to assume $b_2^+(X) > 0$. The situation when $b_2^+(X) = 1$ is very special, because in this case a path of generic parameters cannot always avoid the codimension 1 wall in parameter space, preventing us from proving that the invariants are independent of the choice of parameters. So, to avoid the complications from wall-crossing, we assume $b_2^+(X) \geq 2$ throughout[11].

Given a Spin^c structure $\mathfrak{s}$ and generic parameters $(g, \omega) \in \mathcal{P}$, the moduli space $\mathcal{M}_\omega$ is a smooth, closed, oriented manifold of expected dimension (empty if the expected dimension is negative). Then, the fundamental class of the moduli space defines the Seiberg-Witten invariant of $\mathfrak{s}$.

Theorem 8.1. *For a smooth closed connected oriented 4-manifold X with $b_2^+(X) \geq 2$ the map*

$$\begin{aligned} SW_X : \mathrm{Spin}^c(X) &\longrightarrow H_*(\mathcal{B}^*;\mathbb{Z}) \\ \mathfrak{s} &\longmapsto [\mathcal{M}_\omega] \end{aligned}$$

[11] The case when $b_2^+(X) = 1$ is also interesting, but more complicated, and so we leave it out for time reasons.

is an oriented diffeomorphism invariant of X. That is, if $f : Y \to X$ is an orientation-preserving diffeomorphism, then $f^* \circ SW_X = SW_Y \circ f^*$, i.e. the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Spin}^c(X) & \xrightarrow{SW_X} & H_*(\mathcal{B}^*_X;\mathbb{Z}) \\ {\scriptstyle f^*}\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\downarrow{\scriptstyle f^*} \\ \mathrm{Spin}^c(Y) & \xrightarrow[SW_Y]{} & H_*(\mathcal{B}^*_Y;\mathbb{Z}) \end{array} .$$

Remark 8.2. The commutative square makes sense since $\mathcal{B}^*_X$ is a classifying space for $\mathcal{G}_X$. Any homotopy equivalence between X and Y induces an identification of these classifying spaces, i.e. an induced isomorphism f^*.

Proof of Theorem. We just need to check that SW_X does not depend on the choice of parameters. Since $b_2^+(X) \geq 2$, we can connect two pairs (g,ω), (g',ω') by a generic path disjoint from the wall, which has codimension equal to $b_2^+(X)$. The parametrized moduli space for parameters contained in the path is a compact cobordism or homology between the two moduli spaces. This shows that their homology classes in $\mathcal{B}^*$ are the same. □

Remark 8.3.

(i) $[\mathcal{M}_\omega]$ depends in an uncontrollable way on the orientation of X, in the sense that there is no relation between the invariant of X and that of $\overline{X}$.

(ii) If the moduli space has dimension 0, we obtain a *numerical invariant*.

(iii) One can also work with $\mathcal{M}_\omega$ without considering its orientation and define the associated invariant $SW_X : \mathrm{Spin}^c(X) \to H_*(\mathcal{B}^*;\mathbb{Z}_2)$. This is good enough for some applications.

8.1 The simplest properties

Here are the simplest properties of Seiberg-Witten invariants:

(i) Let $\tau : \mathrm{Spin}^c(X) \to \mathrm{Spin}^c(X)$ be the charge conjugation map. Then $SW_X(\tau(\mathfrak{s})) = \pm SW_X(\mathfrak{s})$. This is because conjugation leads to an identification $(g,\omega) \leftrightarrow (g,-\omega)$. This yields a diffeomorphism (not necessarily oriented) of moduli spaces.

(ii) For any Riemannian metric g on X and Spin^c structure $\mathfrak{s}$ with $SW_X(\mathfrak{s}) \neq 0$ we have

$$2\chi(x) + 3\sigma(X) \leq c_1^2(L_{\mathfrak{s}}) \leq \frac{1}{32\pi^2}\int_X s_g^2 \mathrm{vol}_g .$$

If equality holds in the second inequality, then every solution (A,Φ) of the unperturbed equations with respect to g satisfies $\nabla^A\Phi = F^-_{\hat{A}} = s_g + |\Phi|^2 = 0$.

Proof. The first inequality is equivalent to the statement that the expected dimension of the moduli space for $\mathfrak{s}$ is non-negative.

The assumption $SW_X(\mathfrak{s}) \neq 0$ implies that there must be solutions for every choice of parameters. Taking the parameters $(g, 0)$ and a solution (A, Φ), the second inequality and the statement about the case when it is sharp follows from Corollary 4.31. □

(iii) SW_X has finite support. This is immediate from Corollary 6.7

(iv) If X admits a metric g_0 with $s_{g_0} > 0$, then $SW_X \equiv 0$.

Proof. Suppose $SW_X(\mathfrak{s}) \neq 0$ for some $\mathfrak{s}$. Then there must be solutions of the SW equations for $(\mathfrak{s}, g, \omega)$ for all (g, ω).

Every solution must satisfy $|\Phi|^2 \leq \max\{0, -s_g + 2\sqrt{2}|\omega|\}$. For $\omega = 0$, and $g = g_0$, there can thus only be reducible solutions. For generic (g, ω) near $(g_0, 0)$, there are no reducible solutions and since $-s_{g_0} + 2\sqrt{2}|\omega|$ is negative at $(g_0, 0)$, it is negative on an open neighborhood $U \subset \mathcal{P}$ of $(g_0, 0)$. Hence, for generic $(g, \omega) \in U$, the moduli space is empty. This is a contradiction with the assumption $SW_X(\mathfrak{s}) \neq 0$. □

Remark 8.4.

(i) This fails if $b_2^+(X) = 1$, as exemplified by $\mathbb{C}\mathrm{P}^2$. (In this case one has to specify what one means by the Seiberg-Witten invariant, because of the wall-crossing phenomenon mentioned above.)

(ii) The above argument still works if X admits a metric g with $s_g \geq 0$ and $s_g \not\equiv 0$. This can also be shown by deforming the given metric to one with $s_g > 0$.

(v) In case X admits a scalar-flat metric, we have the following:

Proposition 8.5. *Suppose X admits a metric g with $s_g \equiv 0$. If $2\chi(X) + 3\sigma(X) \geq 0$, then $SW_X(\mathfrak{s}) = 0$ unless $c_1(L_\mathfrak{s})_\mathbb{R} = 0$ and $2\chi(X) + 3\sigma(X) = 0$. In this latter case, $SW_X(\mathfrak{s}) \in H_0(\mathcal{B}^*; \mathbb{Z})$.*

Proof. Suppose X is as in the proposition and $SW_X(\mathfrak{s}) \neq 0$ for some $\mathfrak{s}$. Then on the one hand, since $\dim \mathcal{M}_\omega \geq 0$, we have

$$c_1^2(L_\mathfrak{s}) \geq 2\chi(X) + 3\sigma(X) \geq 0 \,.$$

On the other hand, since $SW_X(\mathfrak{s}) \neq 0$, there must be solutions for $(g, 0)$, where g is the scalar-flat metric. Then we have $|\Phi|^2 \leq -s_g = 0 \implies \Phi \equiv 0 \implies F_{\hat{A}}^+ = \sigma(\Phi, \Phi) \equiv 0$. But then

$$c_1^2(L_\mathfrak{s}) = -\frac{1}{4\pi^2} \int_X \|F_{\hat{A}}^-\|^2 \mathrm{vol}_g \leq 0$$

Thus, $c_1^2(L_\mathfrak{s}) = 2\chi(X)+3\sigma(X) = 0$. Finally, for the solutions $(A, 0)$, we must have $F^-_{\hat{A}} \equiv 0$, so $\hat{A}$ is flat on $L_\mathfrak{s}$ and this implies $c_1(L_\mathfrak{s})_\mathbb{R} = 0$. Since the expected dimension is proportional to $c_1^2(L_\mathfrak{s}) - (2\chi(X) + 3\sigma(X)) = 0$, we see that it vanishes, so $SW_X(\mathfrak{s}) \in H_0(\mathcal{B}^*; \mathbb{Z})$. □

Corollary 8.6. *If X is scalar flat and $2\chi(X) + 3\sigma(X) > 0$, then $SW_X \equiv 0$.*

8.2 Computations of Seiberg-Witten invariants

8.2.1 $K3$ Surfaces and the 4-Torus

Let X be the smooth 4-manifold underlying a complex $K3$ surface, e.g. a smooth degree 4 hypersurface in $\mathbb{C}\mathrm{P}^3$ or the transverse intersection of three quadrics in $\mathbb{C}\mathrm{P}^5$. While there are many different complex structures, it turns out that the smooth manifold X is unique, because all $K3$ surfaces are deformation-equivalent as complex manifolds. By Yau's solution of the Calabi conjecture, every $K3$ surface carries a Ricci-flat Kähler metric, called a Calabi-Yau metric.

The manifold X is oriented by the complex structure and for this orientation $\sigma(X) = -16$, $\chi(X) = 24$. Note that $2\chi(X) + 3\sigma(X) = 0$, so we may still have a non-trivial SW invariant. However, $\bar{X}$ has $\sigma(X) = 16$, $\chi(X) = 24$, so $2\chi(\bar{X}) + 3\sigma(\bar{X}) = 96 > 0$. By the above corollary, $SW_{\bar{X}} \equiv 0$.

The proof of proposition 8.5 shows that, since $\pi_1(K3) = 1$ and hence there is no torsion, $SW_X(\mathfrak{s}) = 0$ unless $c_1(L_\mathfrak{s}) = 0$, i.e. unless $\mathfrak{s}$ is the Spin^c structure induced by the unique Spin structure (uniqueness follows from remark 2.32). Equip X with this Spin^c structure $\mathfrak{s}$ and consider the SW equations for $(g, 0) \in \mathcal{P}$, where g is scalar-flat (for example, take g to be Calabi-Yau). For every solution, $\Phi \equiv 0$, and $\hat{A}$ is flat. In fact $\hat{A}$ is unique up to gauge equivalence, and so the moduli space is a point. We cannot yet conclude that $SW_X(\mathfrak{s}) = \pm 1$ since we need to perturb to a transversal situation (i.e. generic parameters).

At the reducible solution, the linearized SW equations uncouple into the half-de Rham complex

$$\Omega^0(X) \xrightarrow{\mathrm{d}} \Omega^1(X) \xrightarrow{\mathrm{d}^+} \Omega^2_+(X)$$

which has cohomology groups $H^0_{\mathrm{hdR}} \cong \mathbb{R}$, $H^1_{\mathrm{hdR}} = 0$, $H^2_{\mathrm{hdR}} \cong \mathbb{R}^{b_2^+(X)} = \mathbb{R}^3$, and the Dirac operator. Since $\hat{A}$ is flat, the Weitzenböck formula shows that

$$D_A^- D_A^+ = \nabla_A^* \nabla_A + \frac{1}{4} s_g \, ,$$

where the last term on the right hand side also vanishes, because g is scalar-flat. The usual integration by parts now shows that elements of $\ker D_A^+$ are parallel, positive spinors. Similarly, $\operatorname{coker} D_A^+ = \ker D_A^-$ is the space of parallel, negative spinors. A parallel spinor which is nonzero at some point is in fact nonzero everywhere. However, the Euler class $e(V_-)$ does not vanish in this case, hence V_- has no nowhere-vanishing section, i.e. the space of parallel, negative spinors consists only of the zero section and D_A^+ is surjective. The Atiyah-Singer index theorem shows that $\operatorname{ind}_{\mathbb{C}} D_A^+ = -\frac{1}{8}\sigma(X) = 2$. Now $\ker D_A^+ = \mathbb{C}^2$ has a basis of two parallel sections for ∇_A. These two parallel sections give a global trivialization of V_+. The complex

$$0 \longrightarrow \Gamma(V_+) \xrightarrow{D_A^+} \Gamma(V_-)$$

has cohomology $H^1_D \cong \mathbb{C}^2$ and $H^0_D = H^2_D = 0$. We conclude that the elliptic complex has cohomology

$$H^0_{(A,0)} \cong \mathbb{R} \qquad H^1_{(A,0)} \cong \mathbb{C}^2 \qquad H^2_{(A,0)} \cong \mathbb{R}^3 \, .$$

The stabilizer S^1 of the reducible solution in the gauge group acts on $\mathbb{C}^2$ by the standard action and trivially on $\mathbb{R}^3 = \mathbb{C} \oplus \mathbb{R}$. The S^1-equivariant Kuranishi map can be put in the form

$$\begin{aligned} \psi : \mathbb{C}^2 = H^1_{(A,0)} &\longrightarrow H^2_{(A,0)} = \mathbb{C} \oplus \mathbb{R} \\ (z,w) = \Phi &\longmapsto \sigma(\Phi,\Phi) = (z\bar{w}, |z|^2 - |w|^2) \, . \end{aligned}$$

Recall that a neighbourhood of $[A,0] \in \mathcal{M}_0$ is of the form $\psi^{-1}(0)/S^1$. Let us perturb the curvature equation by $\omega \in \mathcal{H}^2_+$ and consider $\psi^{-1}(\omega)/S^1$. The map ψ is the *cone on the Hopf map*, i.e. restricting to elements satisfying $|\omega|^2 = 1$ we find that $\psi^{-1}(\omega)$ is a Hopf circle in $S^3 \subset \mathbb{C}^2$, and $\psi^{-1}(\omega)/S^1$ is a point. For a generic ω, this unique solution up to gauge for the SW equation with parameters $(g, -\omega)$ is transverse, so $SW_X(\mathfrak{s}) = \pm 1$. In summary, we have proven the following:

Proposition 8.7. *If X is the smooth manifold underlying a $K3$ surface with orientation induced by the complex structure, then*

$$SW_X(\mathfrak{s}) = \begin{cases} \pm 1, & \textit{if } \mathfrak{s} \textit{ is induced by the unique } \mathrm{Spin} \textit{ structure,} \\ 0, & \textit{otherwise.} \end{cases}$$

Corollary 8.8. *There is no orientation-preserving diffeomorphism $K3 \to \overline{K3}$.*

This is already clear form the fact that $\sigma(K3) \neq 0$.

Now, we discuss the four-torus $X = T^4 = \mathbb{R}^4/\mathbb{Z}^4$ with a flat metric g. Some of the arguments that follow have already appeared in Exercise 1 on Sheet 9, Exercise 3 on Sheet 10, and Exercise 2 on Sheet 11.

Since $b_2^+(T^4) = 3$, SW_{T^4} is well-defined. Since $2\chi(X) + 3\sigma(X) = 0$, $SW_{T^4}(\mathfrak{s}) = 0$ and there is no torsion, $SW_{T^4}(\mathfrak{s}) = 0$ unless $\mathfrak{s}$ is induced by a Spin structure (and in fact every Spin structure on T^4 induces the same Spin^c structure, cf. remark 2.32). For parameters $(g, 0)$ with g scalar flat, we saw that all solutions are reducible with $\hat{A}$ flat. The gauge equivalence classes of flat connections make up $H^1_{\mathrm{dR}}(X)/H^1(X;\mathbb{Z}) \cong T^4$. In this case, $\dim \mathcal{M}_0 = 4$ while the expected dimension is 0. It turns out that using the Kuranishi method for T^4 is rather cumbersome, so we use a different technique, which in fact also applies to $K3$ surfaces.

For any smooth closed connected and oriented Riemannian 4-manifold (X, g) which is scalar-flat with $2\chi(X) + 3\sigma(X) = 0$, consider $\mathfrak{s}$ induced by a Spin structure, i.e. $c_1(L_\mathfrak{s}) = 0$. Pick a parallel self-dual 2-form ω (for example the Kähler form of the flat metric on T^4) on (X, g) and consider the SW equations for $\mathfrak{s}$ with parameters (g, ω). Using a solution (A, Φ) we have

$$0 = \langle c_1(L_\mathfrak{s}) \smile [\omega], [X] \rangle = \int_X \frac{i}{2\pi} F_{\hat{A}} \wedge \omega = \int_X \frac{i}{2\pi} F^+_{\hat{A}} \wedge \omega$$

where we used self-duality of ω. Then the above equation, combined with the Weitzenböck formula, yields:

$$\begin{aligned} 0 &= \int_X \langle D^-_A D^+_A \Phi, \Phi \rangle d\mathrm{vol}_g = \int_X \Big(|\nabla^A \Phi|^2 + \frac{1}{2} \langle \gamma(F^+_{\hat{A}}) \Phi, \Phi \rangle \Big) d\mathrm{vol}_g \\ &= \int_X \Big(|\nabla^A \Phi|^2 + 2 \langle F^+_{\hat{A}}, \sigma(\Phi, \Phi) \rangle \Big) d\mathrm{vol}_g = \int_X \Big(|\nabla^A \Phi|^2 + 2 \langle F^+_{\hat{A}}, F^+_{\hat{A}} - \omega \rangle \Big) d\mathrm{vol}_g \\ &= \int_X \Big(|\nabla^A \Phi|^2 + 2 |F^+_{\hat{A}}|^2 \Big) d\mathrm{vol}_g \, . \end{aligned}$$

Hence, Φ is parallel and $F^+_{\hat{A}} \equiv 0$. Since $c_1^2(L_\mathfrak{s})_\mathbb{R} = 0$, we conclude that $F^-_{\hat{A}} = 0$ as well, so $\hat{A}$ is flat. Notice that $\Phi \not\equiv 0$ since $\sigma(\Phi, \Phi) = -\omega$ by the curvature equation. Another linearly independent parallel spinor is given by $J(\Phi)$, where J is the charge conjugation map. Since V_+ admits a trivialization of V_+ by parallel sections, A is a product connection. Such connections are unique up to gauge. The map

$$\begin{aligned} \mathbb{C}^2 = \ker D^+_A &\longrightarrow \mathcal{H}^2_+ \\ \Phi &\longmapsto \sigma(\Phi, \Phi) \end{aligned}$$

has $\sigma^{-1}(-\omega) \cong S^1$ with the constant gauge transformations acting freely on this circle, since $\sigma(\lambda\Phi, \lambda\Phi) = \sigma(\Phi, \Phi)$ if and only if $\lambda \in S^1$. Hence, the solution (A, Φ) is unique up to gauge. Replacing ω by $r\omega$ for some generic $r \in \mathbb{R}$, we can make $f_{r\omega}$ transverse. We have therefore shown the following.

Proposition 8.9. *The SW invariant for the torus T^4 is given by*

$$SW_{T^4}(\mathfrak{s}) = \begin{cases} \pm 1, & \textit{for } \mathrm{Spin}^c \textit{ structures induced by } \mathrm{Spin} \textit{ structures,} \\ 0, & \textit{otherwise.} \end{cases}$$

Corollary 8.10. *$K3$ and T^n, for $n \leq 4$, do not admit metrics with positive scalar curvature.*

Proof. For $K3$ or T^4 this follows from $SW_X \not\equiv 0$. For T^n with $n < 4$, the claim follows by taking products, since a product of a positive scalar curvature metric with a flat metric has positive scalar curvature. □

Remark 8.11. For $K3$, this follows already from the fact that $K3$ is Spin and has nonzero signature; see Theorem 4.21. In the case of T^2, the corollary follows from the Gauss-Bonnet theorem:

$$0 = \chi(T^2) = \frac{1}{2\pi} \int_{T^2} K d\mathrm{vol}_g \,.$$

The conclusion about non-existence of positive scalar curvature metrics is also true for tori of dimension ≥ 5, but the proof is quite different.

8.2.2 Einstein manifolds

Instead of Ricci-flat Calabi-Yau manifolds, we now consider manifolds admitting more general Einstein metrics.

Definition 8.12 (Einstein metric)**.** A Riemannian metric is called Einstein if $\mathrm{Ric}_g = \lambda g$ for some $\lambda \in \mathbb{R}$ or, equivalently, if the trace-free part of the Ricci tensor, $\mathrm{Ric}_{g,0}$, vanishes.

By taking the trace and using that the trace of g is constant, we see that an Einstein metric always has constant scalar curvature. The following are some examples of Einstein 4-manifolds.

Example 8.13.

- Spaces of constant curvature: S^4/Γ, $\mathbb{R}^4/\Gamma \cong T^4/\Gamma'$, $\mathbb{H}^4/\Gamma$, where Γ, Γ' are discrete groups acting freely (and properly discontinuously) by isometries.
- Other locally symmetric spaces: $(\mathbb{C}\mathrm{P}^2, \omega_{\mathrm{FS}})$, $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, $\mathbb{C}\mathbb{H}^2/\Gamma$, $(\mathbb{H}^2 \times \mathbb{H}^2)/\Gamma$.
- Calabi-Yau metrics on $K3$ and finite quotients $K3/\Gamma$.
- Certain Kähler-Einstein metrics with $s_g > 0$ on $\mathbb{C}\mathrm{P}^2 \# k\overline{CP}^2$, for $3 \leq k \leq 8$.
- Kähler-Einstein metrics with $s_g < 0$ on compact complex surfaces with ample canonical bundle. This is a large class of manifolds, which we will discuss below.
- $\mathbb{C}\mathrm{P}^2 \# \overline{\mathbb{C}\mathrm{P}}^2$ with the so-called Page metric.

We use the following result without proof:

Theorem 8.14 (Chern-Gauss-Bonnet in dimension 4)**.** *The Euler characteristic of a compact, connected oriented Riemannian 4-manifold X is given by*

$$\chi(X) = \frac{1}{8\pi^2} \int_X \left(\frac{1}{24} s_g^2 + \|W_+\|^2 + \|W_-\|^2 - \| \mathrm{Ric}_{g,0} \|^2 \right) d\mathrm{vol}_g$$

where $W = W_+ + W_-$ is the Weyl tensor.

As a corollary, we have:

Proposition 8.15 (Berger)**.** *If a 4-manifold X is Einstein, then $\chi(X) \geq 0$ with equality only if any Einstein metric is flat.*

We see that $S^1 \times S^3$ and $T^4 \# T^4$ do not admit Einstein metrics. In the second example the Euler characteristic is negative. In the case of $S^1 \times S^3$ the Euler characteristic is zero, but the manifold cannot have a flat metric.

Proposition 8.16 (Thorpe)**.** *If X^4 is Einstein, then $\chi(X) \geq 3|\sigma(X)|/2$.*

Proof. For this, we need a curvature expression for the first Pontryagin number:

$$\sigma(X) = \frac{1}{3}\langle p_1(X), [X]\rangle = \frac{1}{12\pi^2}\int_X \left(|W_+|^2 - |W_-|^2\right) d\mathrm{vol}_g \; .$$

The Chern-Gauss-Bonnet theorem then yields

$$\begin{aligned} 2\chi(X) &= \frac{1}{4\pi^2}\int_X \left(\frac{1}{24}s_g^2 + |W_+|^2 + |W_-|^2\right) d\mathrm{vol}_g \\ &\geq \left|\frac{1}{4\pi^2}\int_X \left(|W_+|^2 - |W_-|^2\right) d\mathrm{vol}_g\right| \\ &= 3|\sigma(X)| \; . \end{aligned}$$

□

Remark 8.17. Something more can be said. In the case of equality, any Einstein metric must have $s_g \equiv 0$ and either $W_+ \equiv 0$ or $W_- \equiv 0$. Picking an orientation, we may assume $W_+ \equiv 0$, i.e. $2\chi(X) = -3\sigma(X)$. If W_- also vanishes, then X is flat and therefore, by a theorem of Bieberbach, a quotient of T^4. If not, it is locally Calabi-Yau and locally isometric to $K3$, i.e. X is always a quotient of either T^4 or $K3$ by a finite group acting freely by isometries. This characterization of the equality case in Thorpe's inequality is due to Hitchin.

Thorpe's inequality implies that Einstein 4-manifolds satisfy the $11/8$-Conjecture 3.17.

Corollary 8.18. *If X^4 is* Spin *and admits an Einstein metric, then $b_2(X) \geq \frac{11}{8}|\sigma(X)|$.*

Proof. Assume $\sigma(X) \neq 0$; by Rohlin's theorem, $|\sigma(X)| \geq 16$. We compute:

$$\begin{aligned} 8b_2(X) &= 8(\chi(X) - 2 + 2b_1(X)) \\ &\geq 8 \cdot \frac{3}{2}|\sigma(X)| - 16 = 12|\sigma(X)| - 16 = 11|\sigma(X)| + (|\sigma(X)| - 16) \\ &\geq 11|\sigma(X)| \; . \end{aligned}$$

□

Finally, we can say something about the SW invariants of Ricci-flat Einstein manifolds which are not $K3$ or T^4:

Proposition 8.19. *Suppose X is a smooth connected compact oriented 4-manifold with $b_2^+(X) \geq 2$. If X admits a Ricci-flat metric and is not $K3$ or T^4, then $SW_X \equiv 0$.*

Proof. Hitchin's characterization of Einstein manifolds with $2\chi(X) = 3|\sigma(X)|$ shows that such manifolds are (quotients of) $K3$ or T^4. One can explicitly check that the nontrivial quotients of $K3$ and T^4 do not satisfy the above assumptions. Therefore, we only have to consider $2\chi(X) > 3|\sigma(X)|$. Thus $2\chi(X) + 3\sigma(X) \geq 2\chi(X) - 3|\sigma(X)| > 0$. This, together with the fact that X is scalar-flat, yields $SW_X \equiv 0$ by corollary 8.6. □

We can compare the results of Thorpe and Hitchin with the following:

Theorem 8.20 (LeBrun). *Assume X is closed and oriented with $b_2^+(X) \geq 2$ and $SW_X \neq 0$. If X admits an Einstein metric, then $\chi(X) \geq 3\sigma(X)$, with equality only if every Einstein metric is flat, or $X = \mathbb{C}H^2/\Gamma$ (a quotient of the complex hyperbolic ball) and every Einstein metric is a rescaling of the standard (Bergman) metric.*

Proof. Let g be the Einstein metric and $\mathfrak{s}$ a Spin^c structure such that $SW_X(\mathfrak{s}) \neq 0$. Then there must be solutions (A, Φ) to the SW equations for parameters $(g, 0)$. Since $\dim^{\exp}\mathcal{M}_\omega \geq 0$, $c_1^2(L_\mathfrak{s}) \geq 2\chi(X) + 3\sigma(X)$. At the same time, Chern-Weil theory tells us that

$$\begin{aligned} c_1^2(L_\mathfrak{s}) &= \frac{1}{4\pi^2}\int_X (|F_{\hat{A}}^+|^2 - |F_{\hat{A}}^-|^2) d\mathrm{vol}_g \\ &\leq \frac{1}{4\pi^2}\int_X |F_{\hat{A}}^+|^2 d\mathrm{vol}_g = \frac{1}{32\pi^2}\int_X |\Phi|^4 d\mathrm{vol}_g \\ &\leq \frac{1}{32\pi^2}\int_X s_g^2 d\mathrm{vol}_g \leq 3\left(\frac{1}{4\pi^2}\int_X \left(\frac{1}{24}s_g^2 + 2|W_-|^2\right) d\mathrm{vol}_g\right) \\ &= 3(2\chi(X) - 3\sigma(X)) \ . \end{aligned}$$

In conclusion, we find

$$6\chi(X) - 9\sigma(X) \geq 2\chi(X) + 3\sigma(X)$$

which is equivalent to $\chi(X) \geq 3\sigma(X)$.

If we have equality, then we see that $\dim \mathcal{M}_0 = 0$, $F_{\hat{A}}^- \equiv 0$, $|\Phi|^2 = -s_g$ and $W_- \equiv 0$. Using corollary 4.31, we also see that $\nabla^A \Phi \equiv 0$. Therefore, the curvature equation $F_{\hat{A}}^+ = \sigma(\Phi, \Phi)$ implies that $F_{\hat{A}}^+$ is also parallel.

Since for $s_g > 0$ one always has $SW_X \equiv 0$, we may assume $s_g \leq 0$. In case $s_g \equiv 0$, we see that $\Phi \equiv 0$, hence $F_{\hat{A}}^+ \equiv 0$. In this case, the Chern-Gauss-Bonnet theorem shows

$$\chi(X) = \frac{1}{8\pi^2}\int_X |W_+|^2 d\mathrm{vol}_g \qquad\qquad \sigma(X) = \frac{1}{12\pi^2}\int_X |W_+|^2 d\mathrm{vol}_g \ .$$

But then$\chi(X) = \frac{3}{2}\sigma(X)$, but at the same time $\chi(X) = 3\sigma(X)$, hence $\chi(X) = \sigma(X) = 0$. Then $W_+ \equiv 0$, hence all components of the Riemann tensor vanish and (X, g) is flat.

Now assume $s_g < 0$. Then $\Phi \not\equiv 0$ and $F^+_{\hat{A}}$ is a parallel, self-dual 2-form, which is up to scaling a Kähler form for g. But then g is a Kähler metric (to be discussed in the next section), i.e. (X, g) is Kähler-Einstein with $s_g < 0$ and $W_- \equiv 0$. In this situation, it is a general fact that W_+ is parallel. Therefore, the Riemann tensor is parallel and therefore this manifold is a *locally symmetric space*, which means that its universal Riemannian cover is a symmetric space. The universal Riemannian cover is a non-compact, Hermitian symmetric space and by the classification of symmetric spaces, it must be either $\mathbb{C}H^2$ or $H^2 \times H^2$. Since $\chi(X) = 3\sigma(X) > 0$ by Chern-Gauss-Bonnet, we can rule out $H^2 \times H^2$. □

Remark 8.21.

(i) In the last step of this proof, we make use of Hirzebruch's *proportionality principle*, which asserts that (certain) characteristic numbers of locally symmetric spaces like the above are proportional to those of a certain "compact dual" space which is naturally associated to them. This correspondence associates $\mathbb{C}\mathrm{P}^2$ to $\mathbb{C}H^2$ and its quotients, which therefore always satisfy $\chi = 3\sigma$. Similarly, $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ corresponds to $H^2 \times H^2$ and its quotients, which therefore have $\chi > 0$, $\sigma = 0$.

(ii) Comparing this with the results by Hitchin-Thorpe, we see that if the manifold is oriented such that $\sigma(X) \geq 0$, the non-vanishing of SW invariants imposes a stronger topological constraint than the Einstein assumption alone. However, since the SW invariants depend strongly on the orientation (requiring their non-triviality typically *fixes* the orientation), the result of LeBrun sometimes yields no extra information over Hitchin-Thorpe.

8.2.3 Complex surfaces with Kähler-Einstein metrics

In this section, we will describe the SW invariants of one of the most important classes of Einstein manifolds, namely compact complex surfaces equipped with Kähler-Einstein metrics with negative scalar curvature. By results of Aubin and Yau, such a metric always exists if the complex structure has ample canonical bundle.

We first introduce some background material. Suppose that J is an almost complex structure on a 4-manifold X, i.e. $J \in \Gamma(\mathrm{End}(TX))$ with $J^2 = -\mathrm{Id}$, which gives TX the structure of a complex vector bundle. Recall that, in this context, the first Pontryagin class of X is given by

$$p_1(X) = c_1^2(X) - 2c_2(X) \implies c_1^2(X) = p_1(X) + 2c_2(X) \, .$$

Then by the signature formula 3.51 and the fact that $c_2(X) = e(X)$ since (TX, J) is of complex rank two, we see that $\langle c_1^2(X), [X] \rangle = 3\sigma(X) + 2\chi(X)$. Since every integral lift of $w_2(X)$ (such as $c_1(X)$) induces a Spin^c-structure, every J must give rise to a Spin^c structure. In conclusion, almost complex manifolds come with a canonical Spin^c structure, $\mathfrak{s}_{\mathrm{can}}$. See Exercise 1 on Sheet 8.

We extend J $\mathbb{C}$-linearly to $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0} \oplus T^{0,1}$, where $T^{1,0}$ (resp. $T^{0,1}$) is the eigenspace of J with eigenvalue $+i$ (resp. $-i$). As a complex vector bundle, $T^{1,0} \cong (TX, J)$ while $T^{0,1} \cong (TX, -J)$. We get an analogous decomposition of $T^*X \otimes_{\mathbb{R}} \mathbb{C}$ and its exterior powers:

$$\Lambda^k(T^*X \otimes_{\mathbb{R}} \mathbb{C}) = \bigoplus_{p+q=r} \left(\Lambda^p(T^{1,0})^* \otimes \Lambda^q(T^{0,1})^*\right) = \bigoplus_{\substack{p+q=k \\ p,q \geq 0}} \Lambda^{p,q}$$

where $\Lambda^{1,0} = (T^{1,0})^*$.

Definition 8.22 (Hermitian metric)**.** A metric on TX (and the related bundles) with respect to which J is orthogonal, i.e. $g(Jv, Jw) = g(v, w)$, is called a Hermitian metric for J.

Given (X, g, J) where g and J are compatible (i.e. g is Hermitian), the two-form ω defined by $\omega(X, Y) = g(JX, Y)$ is called the *fundamental form*. The fundamental 2-form is non-degenerate and prescribes an orientation for X, since its top power is non-vanishing and therefore defines a volume form.

Definition 8.23 (Kähler manifold)**.** The triple (J, g, ω) makes X into a Kähler manifold if $\nabla J = 0$, or equivalently $\nabla\omega = 0$. If $\mathrm{d}\omega = 0$, we call X *almost* Kähler.

Note that $\nabla J = 0$ implies that J is integrable, while $\nabla\omega = 0$ implies that ω is closed. Thus, Kähler manifolds are almost Kähler and come with both a complex and symplectic structure, which are mutually compatible.

Given a Hermitian metric g, we extend the Hodge star operator to the complexified differential forms $* : \Lambda^{p,q} \to \Lambda^{2-q,2-p}$, given by the formula $\alpha \wedge *\bar{\beta} = g(\alpha, \beta)\mathrm{vol}_g$, which makes $*$ into a $\mathbb{C}$-linear map. This allows us to consider the spaces of self-dual and anti self-dual forms:

$$\begin{aligned} \Lambda^2_+ \otimes_{\mathbb{R}} \mathbb{C} &= \mathbb{C}\,\omega \oplus \Lambda^{2,0} \oplus \Lambda^{0,2} \\ \Lambda^2_- \otimes_{\mathbb{R}} \mathbb{C} &= \Lambda^{1,1}_{\perp_\omega} \end{aligned}$$

where we used that $\omega \in \Lambda^{1,1}$ to define the orthogonal complement $\Lambda^{1,1}_{\perp_\omega} \subset \Lambda^{1,1}$. Now, we explicitly define the canonical Spin^c structure defined by the almost complex structure. Its spinor bundles are given by

$$V_+ = \Lambda^{0,0} \oplus \Lambda^{0,2} \qquad\qquad V_- = \Lambda^{0,1}$$

and we define Clifford multiplication by

$$\begin{aligned} \gamma(a) : \quad V_+ &\longrightarrow V_- \\ (\alpha, \beta) &\longmapsto \sqrt{2}(a^{0,1} \wedge \alpha - *(a^{1,0} \wedge *\beta)) \end{aligned}$$

and

$$\begin{aligned} \gamma(a) : \quad V_- &\longrightarrow V_+ \\ \psi &\longmapsto \sqrt{2}\,(-*(a^{1,0} \wedge *\psi), a^{0,1} \wedge \psi) \end{aligned}$$

One has to check that this definition satisfies the properties of a Clifford module; this is left as an exercise. Let $\alpha \in \Omega^{p,q}(X)$ and define the Dolbeault operators $\partial\alpha := (\mathrm{d}\alpha)^{p+1,q}$ and $\bar{\partial}\alpha := (\mathrm{d}\alpha)^{p,q+1}$ by projecting $d\alpha$ to the different summands. Keep in mind that in the almost complex case, we do not have $\mathrm{d} = \partial + \bar{\partial}$ in general. We can still define the adjoint operators with respect to the L^2 metric. Denote them by ∂^* and $\bar{\partial}^*$. Then our definition for Clifford multiplication makes γ into the symbol of the maps

$$\begin{aligned} V_+ &\longrightarrow V_- \\ (\alpha,\beta) &\longmapsto \sqrt{2}(\bar{\partial}\alpha + \bar{\partial}^*\beta) \end{aligned}$$

$$\begin{aligned} V_- &\longrightarrow V_+ \\ \psi &\longmapsto \sqrt{2}(-\bar{\partial}^*\psi, -\bar{\partial}\psi) \end{aligned}$$

This canonical Spin^c structure $\mathfrak{s}_{\text{can}}$ gives us a way to identify $\mathrm{Spin}^c(X)$ with $H^2(X;\mathbb{Z})$ via the map $H^2(X;\mathbb{Z}) \ni E \mapsto \mathfrak{s}_{\text{can}} \otimes E =: \mathfrak{s}_E$, where we abuse notation to identify a line bundle E with its first Chern class (we will sometimes also "additive" notation in place of tensor products, e.g. $K - E$ may denote the bundle $K \otimes E^{-1}$). The spinor bundles of $\mathfrak{s}_E$ are given by $V_+ = E \oplus (\Lambda^{0,2} \otimes E) = E \oplus (K^{-1} \otimes E)$, where $K := \Lambda^{n,0}$ is the canonical line bundle, and $V_- = \Lambda^{0,1} \otimes E$. However, this choice of reference point is not an oriented diffeomorphism invariant, thus the identification is not natural under pullback.

Lemma 8.24. *If (J, g, ω) is Kähler-Einstein, then*

$$\langle c_1^2(X,J), [X] \rangle = \frac{1}{32\pi^2} \int_X s_g^2 \mathrm{dvol}_g \ .$$

Proof. Let ρ be the curvature 2-form of the connection on the characteristic line bundle $\det V_\pm = K^{-1}$ induced by the Levi-Civita connection. We will need the following fact: if g is Kähler Einstein, then $\rho = i\lambda\omega$ for constant λ. In the present case, $\lambda = s_g/4$ (this can be computed from Chern-Gauss-Bonnet). Since $c_1(X,J) = -c_1(K)$, we have

$$\begin{aligned} \langle c_1^2(X,J), [X] \rangle = \langle c_1^2(K), [X] \rangle &= \int_X \frac{i}{2\pi}\rho \wedge \frac{i}{2\pi}\rho \\ &= \int_X \left(\frac{-s_g}{8\pi}\omega\right) \wedge \left(\frac{-s_g}{8\pi}\omega\right) = \frac{1}{64\pi^2} \int_X s_g^2 \omega \wedge \omega \ . \end{aligned}$$

Finally, note that $\omega^2 = 2d\mathrm{vol}_g$ to obtain the promised result. □

Assume X has a Kähler-Einstein structure (J, g, ω) and E is a complex line bundle such that $SW_X(\mathfrak{s}_E) \neq 0$. Then the SW equations for $\mathfrak{s}_E$ and $(g, 0) \in \mathcal{P}$ must have solutions (A, Φ). We have

$$\langle c_1^2(X,J),[X]\rangle = 2\chi(X)+3\sigma(X) \leq \langle c_1^2(L_{\mathfrak{s}_E}),[X]\rangle = \frac{1}{4\pi^2}\int_X \left(|F_{\hat{A}}^+|^2 - |F_{\hat{A}}^-|^2\right)\mathrm{vol}_g$$
$$\leq \frac{1}{4\pi^2}\int_X |F_{\hat{A}}^+|^2\mathrm{vol}_g = \frac{1}{32\pi^2}\int_X |\Phi|^4\mathrm{vol}_g \leq \frac{1}{32\pi^2}\int_X s_g^2\mathrm{vol}_g = \langle c_1^2(X,J),[X]\rangle\ ,$$

where the last inequality is from the lemma. Hence, all the inequalities are equations and as a consequence $\dim\mathcal{M}_0 = 0$, $F_{\hat{A}}^- \equiv 0$ and $|\Phi|^4 = s_g^2 =$ const. We have the following cases.

- If $s_g > 0$, and $b_2^+(X) \geq 2$, then $SW_X \equiv 0$. In fact, this case does not occur since it turns out that $b_2^+(X) = 1$ for a Kähler-Einstein surface X with $s_g > 0$.
- If $s_g \equiv 0$, then g is flat or Calabi-Yau, i.e. either T^4 or $K3$.
- For $s_g < 0$, we see that $|\Phi|^2 = -s_g > 0$, so the solution is irreducible. Recall from corollary 4.31 that if the Spin^c structure $\mathfrak{s}$ admits solutions for parameters $(g,0)\in\mathcal{P}$, then

  $$\langle c_1^2(L_{\mathfrak{s}}),[X]\rangle \leq \frac{1}{32\pi^2}\int_X s_g^2\mathrm{vol}_g$$

  with equality if and only if $F_A^- = 0$, $\nabla_A\Phi = 0$, and $|\Phi|^2 = -s_g$. Hence, Φ is non-zero and parallel; and it trivializes either E or $K^{-1}\otimes E$. In the first case, $\mathfrak{s}_E = \mathfrak{s}_{\text{can}}$. In the second case, $\mathfrak{s}_E = \mathfrak{s}_{\text{can}}\otimes K = \overline{\mathfrak{s}_{\text{can}}}$, since K is the characteristic line bundle. These are the only Spin^c-structures for which $SW_X(\mathfrak{s}_E)$ may be non-zero. In both cases, there are in fact tautological solutions, unique up to gauge.

The above discussion can be summarized as follows:

Theorem 8.25 (Witten)**.** *If X is a complex surface with $b_2^+(X) \geq 2$, which admits a Kähler-Einstein metric with $s_g < 0$, then*

$$SW_X(\mathfrak{s}) = \begin{cases} \pm 1 & \textit{if } \mathfrak{s} = \mathfrak{s}_{can} \textit{ or } \bar{\mathfrak{s}}_{can} \\ 0 & \textit{otherwise.}\end{cases}$$

Remark 8.26. In the situation of the theorem above, suppose that $f : X \to X$ is an orientation-preserving diffeomorphism. Then f^* maps $\mathfrak{s}_{\text{can}}$ either to itself or to its conjugate. For the action of f^* on $H^2(X;\mathbb{Z})$, this implies that $f^*c_1(X,J) = \pm c_1(X,J) \neq 0$, or equivalently $f^*K = \pm K$ (denoting the first Chern class of the canonical bundle by K).

Our computations so far can be summarized as follows:

Theorem 8.27. *Let X be one of the following Kähler 4-manifolds*

- $X = T^4$,

- $X = K3$,
- X *Kähler-Einstein with* $s_g < 0$ *and* $b_2^+ \geq 2$.

Then $SW_X(\mathfrak{s}) = \pm 1$ *if* $\mathfrak{s} = \mathfrak{s}_{can}$ *or* $\mathfrak{s} = \bar{\mathfrak{s}}_{can}$, *and* $SW_X(\mathfrak{s}) = 0$ *otherwise.*

8.2.4 Further generalizations

The above calculations can be generalized quite substantially. One can calculate completely all the Seiberg-Witten invariants of Kähler surfaces, without the assumption that they admit a Kähler-Einstein metric. In the case of so-called minimal surfaces of general type[12] Theorem 8.25 remains true, in the sense that the Seiberg-Witten invariant is ± 1 for the canonical Spin^c structure and for its conjugate, and is zero otherwise. As above, this leads to the conclusion that the canonical class is an oriented diffeomorphism invariant (up to sign).

Going yet further, one can relax the Kähler condition and consider arbitrary symplectic manifolds. In this case again, some part of the above theorems remains true. For every symplectic structure there is a unique homotopy class of compatible almost complex structures, and this again gives rise to a canonical Spin^c structure. By a theorem of Taubes, the Seiberg-Witten invariants of the canonical Spin^c structure and of its conjugate are again ± 1. One can also prove some constraints on other Spin^c structures with non-trivial Seiberg-Witten invariants, but in general these other invariants are also non-trivial. As we saw in section 8.1 the number of Spin^c structures with non-trivial Seiberg-Witten invariant is always finite. However, even in the case of symplectic manifolds, this number can be arbitrarily large for suitably chosen manifolds.

8.3 Applications

8.3.1 Immediate applications

Since the Seiberg-Witten invariants are preserved by orientation-preserving diffeomorphisms, their most obvious application is to proving that certain manifolds cannot be diffeomorphic to each other, because they have different invariants. This has been done in many cases, showing that certain manifolds which are homeomorphic by Freedman's theorem 3.15 are in fact not diffeomorphic. One can even show that many closed simply connected 4-manifolds have infinitely many distinct smooth structures distinguished by their Seiberg-Witten invariants.

Such applications sometimes use complete calculations of Seiberg-Witten invariants, such as the ones for Kähler surfaces which we discussed above. More often though, the applications rely not on complete calculations, but on weaker information. For example, if one manifold has identically zero Seiberg-Witten invariant, and the other one does not, then they cannot be diffeomorphic, without any need to do a complete calculation in the second case. For this kind of argument one would like to have easy to verify vanishing and non-vanishing results for the invariants.

Concerning vanishing results, we saw in section 8.1 that the Seiberg-Witten invariants of a connected closed oriented smooth 4-manifold X with $b_2^+(X) \geq 2$ vanish if X admits a Riemannian

[12]This refers to the so-called Enriques-Kodaira classification of compact complex surfaces.

metric of positive scalar curvature. Another important vanishing result is the following theorem, which we will not prove:

Theorem 8.28. *If a connected closed oriented smooth* 4*-manifold* X *with* $b_2^+(X) \geq 2$ *is diffeomorphic to a smooth connected sum* $X_1 \# X_2$ *with* $b_2^+(X_i) > 0$ *for both* $i = 1, 2$*, then its Seiberg-Witten invariants vanish identically.*

Example 8.29. The Seiberg-Witten invariants of $p\mathbb{C}P^2 \# q\overline{\mathbb{C}P^2}$ with $p \geq 2$ vanish identically.

Beyond proving that certain homeomorphic manifolds are not diffeomorphic to each other, the Seiberg-Witten invariants have many other applications, some of which arise from the connections between the Seiberg-Witten equations and various geometric structures. For example, the invariants often provide obstructions to the existence of a special geometric structure, for example a metric of positive scalar curvature, or a symplectic structure. The invariants also show that certain geometric structures are incompatible, as in the following:

Theorem 8.30. *A connected closed symplectic* 4*-manifold* X *with* $b_2^+(X) \geq 2$ *cannot admit a Riemannian metric of positive scalar curvature.*

This is clear since in the symplectic case the invariant is non-trivial, whereas in the psc case it is trivial.

Finally, let us note, again without proof, that there are some obvious relations between the Seiberg-Witten invariants of X and those of $X \# \overline{\mathbb{C}P^2}$. This should not be surprising, since if X is a complex surface, then $X \# \overline{\mathbb{C}P^2}$ is the smooth manifold underlying the blowup of X at a point. If X is Kähler, then so is the blowup, and the calculations of Seiberg-Witten invariants can be carried out on both of them. In general, if $\mathfrak{s} \in \mathrm{Spin}^c(X)$ with $c_1(L_{\mathfrak{s}}) = c$, then there is a unique Spin^c structure $\mathfrak{s}' \in \mathrm{Spin}^c(X \# \overline{\mathbb{C}P^2})$ for which

$$c_(L_{\mathfrak{s}'}) = c + E$$

where $E \in H^2(\mathbb{C}P^2; \mathbb{Z}$ is a generator. The name E comes from the fact that in the complex case this generator is the cohomology class of the exceptional divisor of the blowup.

Theorem 8.31. *For every* X *with* $b_2^+(X) \geq 2$*, we have*

$$SW_{X \# \overline{\mathbb{C}P^2}}(\mathfrak{s}') = SW_X(\mathfrak{s}) \ .$$

In particular, if the Seiberg-Witten invariant of X is non-trivial, so is that of $X \# \overline{\mathbb{C}P^2}$.

8.3.2 The adjunction inequality for embedded surfaces

We saw in Theorem 3.47 that if X is a closed, (almost) complex 4-manifold and Σ is an embedded complex curve equipped with the complex orientation, then

$$2g(\Sigma) - 2 = \Sigma \cdot \Sigma + K \cdot \Sigma \ ,$$

where $K = -c_1(X)$ is the canonical class of X. We now want to prove that often there cannot be any smoothly embedded surface in the same homology class whose genus is smaller than that of a holomorphic representative.

In the presence of non-vanishing SW invariants, we obtain the following lower bound for the genus of a smoothly embedded surface:

Theorem 8.32 (Adjunction Inequality for Surfaces)**.** *Let X be a connected closed oriented 4-manifold with $b_2^+(X) \geq 2$, and $\Sigma \subset X$ a smoothly embedded, oriented and connected surface with $g(\Sigma) \neq 0$ and $\Sigma \cdot \Sigma \geq 0$. If $SW_X(\mathfrak{s}) \neq 0$, then*

$$2g(\Sigma) - 2 \geq \Sigma \cdot \Sigma + |c_1(L_\mathfrak{s}) \cdot \Sigma| \, .$$

This inequality gives us a lower bound on the genus $g(\Sigma)$ depending only on the homology class represented by the surface, as long as this homology calls is of non-negative self-intersection.

Corollary 8.33. *Let X be a Kähler 4-manifold with $b_2^+(X) \geq 2$. Suppose $\Sigma \subset X$ is a complex curve with $\Sigma \cdot \Sigma \geq 0$ and let Σ' be a connected, smoothly embedded surface with $[\Sigma'] = [\Sigma]$. Then $g(\Sigma') \geq g(\Sigma)$.*

Proof. Since X is Kähler and $\Sigma \subset X$ complex, Σ is also Kähler and hence $\langle \omega, [\Sigma] \rangle \neq 0$, i.e. $[\Sigma] \neq 0$ and is of infinite order. Now if Σ' is a sphere such that $[\Sigma'] = [\Sigma]$ and $\Sigma \cdot \Sigma \geq 0$, then by the exercises on Sheet 12 we must have $SW_X \equiv 0$, but that is false since we are on a Kähler manifold. Hence $g(\Sigma') > 0$. Now, we use the adjunction inequality for $\mathfrak{s}_{\text{can}}$, since we know that $SW_X(\mathfrak{s}_{\text{can}}) \neq 0$. This yields

$$2g(\Sigma') - 2 \geq \Sigma \cdot \Sigma + |K \cdot \Sigma| \geq \Sigma \cdot \Sigma + K\Sigma = 2g(\Sigma) - 2$$

and therefore $g(\Sigma') \geq g(\Sigma)$. □

Remark 8.34.

(i) The Thom conjecture asks whether complex curves with $\Sigma \cdot \Sigma \geq 0$ in a complex surface have minimal genus among all smooth surfaces in their homology class. The above is a special case. The Thom conjecture was proven in 1994, very soon after the advent of SW theory, by Kronheimer and Mrowka.

(ii) The result actually holds for all symplectic 4-manifold, with the same adjunction formula, and the notion of "complex" replaced with "J-holomorphic" (J is an ω-compatible almost complex structure). It is proven using $c_1(L_{\mathfrak{s}_{\text{can}}}) = -K$.

We start the proof of the adjunction inequality by considering the case $\Sigma \cdot \Sigma = 0$.

Theorem 8.35. *Let X be a closed, oriented, smooth 4-manifold with $b_2^+ \geq 2$. Let $\mathfrak{s} \in \mathrm{Spin}^c(X)$ with $SW_X(\mathfrak{s}) \neq 0$. If $\Sigma \subset X$ is a smoothly embedded connected surface with $g(\Sigma) \neq 0$ and*

$\Sigma \cdot \Sigma = 0$, *then*

$$2g(\Sigma) - 2 \geq |c_1(L_{\mathfrak{s}}) \cdot \Sigma| \ .$$

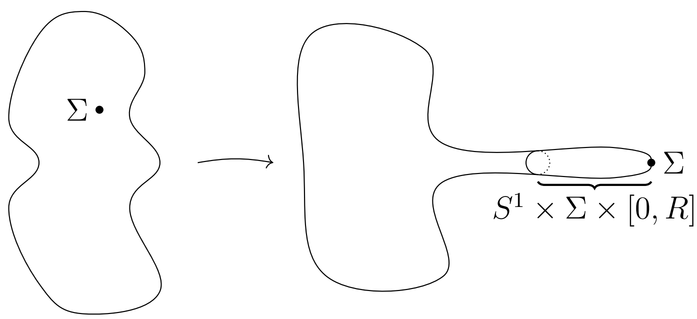
Proof. Let g be a Riemannian metric on X and $\mathfrak{s} \in \mathrm{Spin}^c(X)$ such that $SW_X(\mathfrak{s}) \neq 0$. There must exist solutions for parameters $(g, 0)$. Observe that

$$\frac{1}{4\pi^2}\int_X |F_{\hat{A}}|^2 d\mathrm{vol}_g = \frac{1}{2\pi^2}\int_X |F_{\hat{A}}^+|^2 d\mathrm{vol}_g - c_1^2(L_{\mathfrak{s}})$$

and by the monopole equations this becomes

$$\begin{aligned} 1\frac{1}{4\pi^2}\int_X |F_{\hat{A}}|^2 d\mathrm{vol}_g &= \frac{1}{2\pi^2}\int_X |\sigma(\Phi,\Phi)|^2 d\mathrm{vol}_g - c_1^2(L_{\mathfrak{s}}) \\ &= \frac{1}{16\pi^2}\int_X |\Phi|^4 d\mathrm{vol}_g - c_1^2(L_{\mathfrak{s}}) \\ &\leq \frac{1}{16\pi^2}\int_X s_g^2 d\mathrm{vol}_g - c_1^2(L_{\mathfrak{s}}) \ . \end{aligned}$$

Choose a metric g_Σ on the embedded surface Σ which is of constant curvature, and such that $\mathrm{vol}_{g_\Sigma}(\Sigma) = 1$. Since $\Sigma \cdot \Sigma = 0$, the normal bundle $\nu(\Sigma)$ is trivial, hence diffeomorphic to $\Sigma \times D^2$. Choose a metric on X such that in $\nu(\Sigma)$ there is a metric cylinder of the form $\Sigma \times S^1 \times [0, R]$ equipped with product metric, given by g_Σ times the standard metrics on S^1 and $[0, R] \subset \mathbb{R}$; additionally, on $X \setminus \nu(\Sigma)$ the metric should not depend on R. Call it g_R.



There exist solutions to the SW equations for parameters $(g_R, 0)$ for any $R > 0$. The curvature satisfies

$$\begin{aligned} \frac{1}{4\pi^2}\int_X |F_{\hat{A}}|^2 d\mathrm{vol}_{g_R} &\geq \int_{\Sigma\times S^1\times[0,R]} \left|\frac{i}{2\pi}F_{\hat{A}}\right|^2 dd\mathrm{vol}_{g_R} \\ &= \int_{S^1\times[0,R]}\int_\Sigma \left|\frac{i}{2\pi}F_{\hat{A}}\right|^2 d\mathrm{vol}_\Sigma \wedge d\mathrm{vol}_{S^1\times[0,R]} \\ &\geq \int_{S^1\times[0,R]} \left(\int_\Sigma \frac{i}{2\pi}F_{\hat{A}}\right)^2 d\mathrm{vol}_{S^1\times[0,R]} \\ &= \int_{S^1\times[0,R]} (c_1(L_{\mathfrak{s}})\cdot\Sigma)^2 d\mathrm{vol}_{S^1\times[0,R]} \\ &= R(c_1(L_{\mathfrak{s}})\cdot\Sigma)^2 \end{aligned}$$

where we discarded part of our integral in the first step, then applied Fubini's theorem and later used the fact that we set $\mathrm{vol}_{S^1} = 1$. To relate this to our earlier inequality in an interesting way, we consider the scalar curvature term:

$$\begin{aligned}\int_X s_{g_R}^2 d\mathrm{vol}_{g_R} &= \int_{\Sigma\times S^1\times[0,R]} s_{g_R}^2 \mathrm{vol}_{g_R} + C\\ &= R\int_\Sigma s_{g_\Sigma}^2 \mathrm{vol}_{g_\Sigma} + C\\ &= R\left(\int_\Sigma s_{g_\Sigma} \mathrm{vol}_{g_\Sigma}\right)^2 + C\\ &= R(4\pi(2-2g(\Sigma)))^2 + C\ .\end{aligned}$$

Here, C is a constant (i.e. independent of R). Since s_{g_Σ} is constant, we may take the square out of the integral, as we did. Then we recall that on a surface, s_g is twice the Gauss curvature and that its integral therefore equals $4\pi\chi(\Sigma)$. This shows that for any $R > 0$

$$R(c_1(L_{\mathfrak{s}})\cdot\Sigma)^2 \le \frac{1}{4\pi^2}\int_X |F_{\hat{A}}|^2 d\mathrm{vol}_{g_R} \le \frac{1}{16\pi^2} s_{g_R}^2 d\mathrm{vol}_{g_R} - c_1^2(L_{\mathfrak{s}}) = R(2g(\Sigma)-2)^2 + C'$$

where the constant C' contains terms that do not depend on R. Since $R > 0$, we have

$$(c_1(L_{\mathfrak{s}})\cdot\Sigma)^2 \le (2g(\Sigma)-2)^2 + \frac{C'}{R}$$

and in the limit $R\to\infty$, we can use the assumption $g(\Sigma) > 0$ to obtain

$$|c_1(L_{\mathfrak{s}})\cdot\Sigma| \le 2g(\Sigma) - 2$$

which is the result we were after. □

In the case $\Sigma\cdot\Sigma > 0$ we use Theorem 8.31 to reduce to the case we have already proved.

Proof of the Adjunction Inequality. Suppose $\Sigma\cdot\Sigma = k > 0$. Since $c_1(L_{\bar{\mathfrak{s}}}) = -c_1(L_{\mathfrak{s}})$ and $SW_X(\bar{\mathfrak{s}}) = \pm SW(\mathfrak{s})$, we may assume that $c_1(L_{\mathfrak{s}})\cdot\Sigma \ge 0$. Instead of $\Sigma\subset X$, consider $\Sigma' = \Sigma\#\overline{\mathbb{C}\mathrm{P}^1} \subset X' = X\#\overline{\mathbb{C}\mathrm{P}^2}$. Since Σ' arises from tubing together Σ and a sphere, its genus is the same, however its homology class is $[\Sigma] - E$, and so $\Sigma'\cdot\Sigma' = k-1$ and $Q_{X'} = Q_X \oplus (-1)$. If we can find a Spin^c structure $\mathfrak{s}'$ on X' such that $SW_X(\mathfrak{s}') \neq 0$ and $c_1(L_{\mathfrak{s}'})\cdot\Sigma' = c_1(L_{\mathfrak{s}})\cdot\Sigma + 1$, then all of our assumptions still hold and we do not spoil the (prospective) adjunction inequality. By induction we can then reduce the proof to the case of a surface of zero selfintersection.

Now Theorem 8.31 gives us what we want, since $c_1(L_{\mathfrak{s}'}) = c_1(L_{\mathfrak{s}}) + E$ and so

$$c_1(L_{\mathfrak{s}'})\cdot\Sigma' = (c_1(L_{\mathfrak{s}}) + E)\cdot(\Sigma - E) = c_1(L_{\mathfrak{s}})\cdot\Sigma + 1\ .$$

This completes the proof. □