Mathematical Gauge Theory 1 Problem Sheets

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Problem Sheet 1

Lie Groups

Exercise 1.1. Let G be a Lie group. Show that the Lie bracket $[X, Y]$ of two left-invariant vector fields X, Y is a left-invariant vector field.

Exercise 1.2. Consider the 3-dimensional sphere S^3 as the set of unit quaternions, i.e.

$$S^3 = \left\{a + ib + jc + kd \in \mathbb{H} \mid a^2 + b^2 + c^2 + d^2 = 1\right\}$$

Show that S^3 is a Lie group.

Exercise 1.3. Consider the Lie group $\mathrm{SL}(2, \mathbb{R})$ and its Lie algebra $\mathfrak{sl}(2, \mathbb{R})$.

(1) Compute $\mathrm{tr}(\exp X)$ for $X \in \mathfrak{sl}(2, \mathbb{R})$.

(2) Show that the exponential map $\exp\colon \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ is not surjective.

Exercise 1.4. Let G be a connected Lie group.

(1) Show that if $H \subset G$ is an open subgroup then $H = G$.

(2) Let $U \subset G$ an open neighbourhood of the identity e. Prove that the set $W = \bigcup\limits_{n=1}^{\infty} U^n$ contains an open subgroup of G. Deduce that $W = G$.

(3) Show that every group element $g \in G$ is of the form $g = \exp X_1 \cdot \exp X_2 \cdot \dots \cdot \exp X_n$ for finitely many vectors $X_1, \dots, X_n$ in the Lie algebra $\mathfrak{g}$ of G.

(4) Let $\phi, \psi : G \to K$ be Lie group homomorphisms. Show that if $\phi_* = \psi_* : \mathfrak{g} \to \mathfrak{h}$ then $\phi = \psi$.

Problem Sheet 2

Fiber Bundles

Exercise 2.1. Suppose $\pi : P \to M$ is a principal G-bundle and let $f : N \to M$ be a smooth map. Define the **pullback** of P under f to be the space

$$f^*P := \{(x, p) \in N \times P \mid f(x) = \pi(p)\}$$

(1) Show that the map

$$\begin{aligned} \pi' : f^*P &\to N \\ (x, p) &\mapsto x \end{aligned}$$

defines a principal G-bundle.

(2) Let $W \subset M$ be an embedded submanifold. Show that the restriction $\pi : \pi^{-1}(W) \to W$ is a well defined principal G-bundle.

(3) Prove that the bundle f^*P is trivial if f is a constant map.

(4) Prove that the bundle f^*P is trivial if P is trivial.

Exercise 2.2. Define the Möbious strip M to be the submanifold

$$M = \left\{ \left(e^{i\theta}, re^{i\theta/2}\right) \in S^1 \times \mathbb{C} \mid \theta \in [0, 2\pi], r \in [-1, 1] \right\}$$

and let $\pi : M \to S^1$ be the projection on the first factor.

(1) Show that $\pi : M \to S^1$ is a fibre bundle with fibre $[-1, 1]$.

(2) Prove that the boundary ∂M is connected and that the bundle $\pi : M \to S^1$ is not trivial.

(3) Prove that the image of any smooth section $s : S^1 \to M$ intersects the zero section $e^{i\theta} \mapsto \left(e^{i\theta}, 0\right)$.

Exercise 2.3. Let $\pi : M \to S^1$ be the fibre bundle from Exercise 2.2 and consider the maps

$$\begin{aligned} f_n : S^1 &\to S^1 \\ e^{i\theta} &\mapsto e^{in\theta} \end{aligned}$$

for $n \in \mathbb{Z}$.

(1) Show that the pull-back bundle f_n^*M is isomorphic to the bundle $\pi_n : M_n \to S^1$ defined by

$$M_n = \left\{ \left(e^{i\theta}, re^{in\theta/2}\right) \in S^1 \times \mathbb{C} \mid \theta \in [0, 2\pi], r \in [-1, 1] \right\}$$

where π_n is the projection on the first factor.

(2) For which $n \in \mathbb{Z}$ is the pullback bundle f_n^*M trivial?

Exercise 2.4.

(1) Let $\pi : E \to M$ be a fiber bundle such that the base M and the fibre F are connected. Show that E is connected.

(2) Show that the $\mathrm{SO}(n)$ principal bundle $\pi : \mathrm{SO}(n + 1) \to S^n$ is the bundle of oriented orthonormal frames of the tangent bundle TS^n.

(3) Use part (1) to show that the group $\mathrm{SO}(n)$ is connected for all n.

Problem Sheet 3

Principle Bundle

Exercise 3.1. Let $\pi : P \to B$ be a principal G-bundle, $\{U_i\}$ an open cover of B by trivializing sets and ω_i1-forms on U_i with values in $\mathfrak{g}$. Show that if

$$\omega_j = \mathrm{Ad}\left(\psi_{ij}^{-1}\right)\omega_i + \theta_{ij} \text{ on } U_i \cap U_j$$

then there is a unique connection 1-form ω on P such that $\omega_i = s_i^* \omega$.

Remark. Here θ_{ij}, ψ_{ij} and s_i are defined as in the lectures.

Exercise 3.2. Let $\pi : P \to B$ be a principal G-bundle and fix a connection $H \subset TP$. Consider vector fields $V, W \in \mathfrak{X}(B)$ and let $\widetilde{V}, \widetilde{W}$ be their horizontal lifts.

(1) Show that $\widetilde{V+W} = \widetilde{V} + \widetilde{W}$.

(2) Show that $\widetilde{fV} = (f \circ \pi)\widetilde{V}$ for $f \in C^\infty(B)$.

(3) Show that $\widetilde{[V,W]} = [\widetilde{V}, \widetilde{W}]_H$.

Exercise 3.3. Let G be a Lie group and $\mathfrak{g} = T_eG$ its Lie algebra. Consider a continuous curve Y_t in T_eG with $t \in [0,1]$. Show that there exist a unique curve a_t in G of class C^1 such that $a_0 = e$ and $\dot{a}_t a_t^{-1} = Y_t$ for all $t \in [0,1]$.

Exercise 3.4. Consider S^3 as the set of unit vectors in $\mathbb{C}^2$. By abuse of notation let $\pi : S^3 \to S^2 = \mathbb{CP}^1$ be the restriction of the projection $\pi : \mathbb{C}^2 \backslash \{0\} \to \mathbb{CP}^1$ to S^3.

(1) Show that $\pi : S^3 \to S^2$ is a principal S^1-bundle (called the Hopf bundle).

(2) Consider S^1 as the unit circle in $\mathbb{C}$ with Lie algebra $i\mathbb{R}$ and exponential map $\exp(Y) = e^{iy}$ where $Y = iy \in i\mathbb{R}$. Define 1-forms on S^3 with values in $\mathbb{C}$ by

$$\alpha_j\left(X_0, X_1\right) = X_j, \quad \bar{\alpha}_j\left(X_0, X_1\right) = \bar{X}_j$$

by using the identification

$$T_{(z_0,z_1)}S^3 = \left\{(X_0, X_1) \in \mathbb{C}^2 \mid \mathcal{R}\left(\bar{z}_0 X_0 + \bar{z}_1 X_1\right) = 0\right\}$$

where $\mathcal{R}(\beta)$ denotes the real part of $\beta \in \mathbb{C}$. Show that the 1-form on S^3

$$A_{(z_0,z_1)} = \frac{1}{2}\left(\bar{z}_0\alpha_0 - z_0\bar{\alpha}_0 + \bar{z}_1\alpha_1 - z_1\bar{\alpha}_1\right)$$

has values in $i\mathbb{R}$ and is a connection 1-form for the Hopf bundle.

Problem Sheet 4

Connection

Exercise 4.1. This is a continuation of Exercise 3.4 on sheet 3. We will use the same notation.

(1) Show that the curvature of the connection 1 -form A on the Hopf bundle is given by

$$\Omega^A = -\left(\alpha_0 \wedge \bar{\alpha}_0 + \alpha_1 \wedge \bar{\alpha}_1\right)$$

(2) Define a 2-form on $\mathbb{C}$ by

$$\widetilde{\Omega}_w := -\frac{1}{\left(1+|w|^2\right)^2} dw \wedge d\overline{w}$$

Let $U_1 := \left\{[z_0, z_1] \in \mathbb{CP}^1 \mid z_1 \neq 0\right\}$ and $\psi_1 : U_1 \to \mathbb{C}$ be given by $[z_0 : z_1] \mapsto z_0/z_1$. Show that $\psi_1^* \widetilde{\Omega}$ can be prolonged to a form $\Omega_{\mathbb{CP}^1}$ on all of $\mathbb{CP}^1$ which satisfies $\pi^* \Omega_{\mathbb{CP}^1} = \Omega^A$. Deduce that for every local section s of π, defined on some open set $V \subset \mathbb{CP}^1$ one has $s^* \Omega_A = \Omega_{\mathbb{CP}}\Big|_V$.

(3) Compute the integral $\displaystyle\int_{\mathbb{CP}^1} \Omega_{\mathbb{CP}^1}$.

Exercise 4.2. Let G be a Lie group and $H \subset G$ a closed subgroup, with Lie algebras $\mathfrak{h} \subset \mathfrak{g}$. We know that $\pi : G \to G/H$ is an H-principal bundle. Assume that there exists a vectorspace complement $\mathfrak{m} \oplus \mathfrak{h} = \mathfrak{g}$ such that $\mathrm{Ad}(H)\mathfrak{m} \subset \mathfrak{m}$.

(1) Consider $\omega = \pi_{\mathfrak{h}} \circ \theta \in \Omega^1(G, \mathfrak{h})$, where θ is the tautological 1-form on G with values in $\mathfrak{g}$. Prove that ω is a connection 1-form on $G \to G/H$.

(2) Show that the vertical and horizontal subspaces defined by ω at a point $g \in G$ are given by $DL_g\mathfrak{h}$ and $DL_g\mathfrak{m}$.

(3) Prove that the curvature of the connection ω is given by

$$\Omega = -\frac{1}{2}\pi_{\mathfrak{h}} \circ [\pi_{\mathfrak{m}} \circ \theta, \pi_{\mathfrak{m}} \circ \theta] \in \Omega^2(G, \mathfrak{h})$$

where the commutator is taken in $\mathfrak{g}$.

Exercise 4.3. Let $P \to B$ be a principal G-bundle. Assume P admits a reduction of the structure group to a closed Lie subgroup $K \subset G$. Prove that P admits a connection with holonomy contained in K.

Problem Sheet 5

Constructions in Gauge Theory

Exercise 5.1. Let G be a Lie group and $\mathrm{Ad} : G \to GL(\mathfrak{g})$ its adjoint representation. Show that the differential of Ad at the identity element is given by

$$\begin{aligned} D_e \,\mathrm{Ad} : \mathfrak{g} &\to \mathrm{End}(\mathfrak{g}) \\ X &\mapsto (Y \mapsto [X, Y]) \end{aligned}$$

Exercise 5.2. Let $\widetilde{B} \to B$ be the universal covering space of a smooth connected manifold B. Regard $\widetilde{B}$ as a principal bundle with fibre the discrete Lie group $\pi_1(B)$ acting on the right. Let G be a Lie group and consider the bundle $\widetilde{B} \times_{\pi_1(B)} G$ associated to $\widetilde{B}$ by $\rho : \pi_1(B) \to G$ with $\pi_1(B)$ acting on G by ρ composed with left multiplication.

(1) Show that $\widetilde{B} \times_{\pi_1(B)} G$ is a principal G-bundle over B which admits a flat connection.

(2) Show that for any representation $\rho : \pi_1(B) \to \mathrm{GL}(n, \mathbb{R})$ the associated vector bundle $\widetilde{B} \times_\rho \mathbb{R}^n$ admits an integrable connection, i.e. an integrable horizontal subbundle $H \subset T(\widetilde{B} \times_\rho \mathbb{R}^n)$.

Exercise 5.3. Let $\pi : E \to B$ be a rank k vector bundle with fiber V over a smooth manifold B.

(1) Prove that the set of bases, or frames, in the fibres of E naturally forms a principal $\mathrm{GL}(V)$-bundle over B (the frame bundle $\mathrm{Fr}(E)$).

(2) Show that the vector bundle $\mathrm{Fr}(E) \times_\rho V$ associated to $\mathrm{Fr}(E)$ by the tautological representation $\rho : \mathrm{GL}(k, \mathbb{R}) \to \mathrm{GL}(V)$ is isomorphic to E.

Exercise 5.4. Let $\pi : P \to B$ be a principal G-bundle.

(1) Consider two representations $\rho, \rho' : G \to \mathrm{GL}(n, \mathbb{R})$. Show that the associated vector bundles $P \times_\rho \mathbb{R}^n$ and $P \times'_\rho \mathbb{R}^n$ are isomorphic if and only if there exists a smooth map $\phi : P \to \mathrm{GL}(n, \mathbb{R})$ satisfying $\rho'_g \phi_{pg} = \phi_p \rho_g$.

(2) Analogously prove that, for two smooth actions $\mu : G \times F \to F$ and $\mu' : G \times F \to F$, the associated fibre bundles $P \times_{G,\mu} F$ and $P \times_{G,\mu'} F$ are isomorphic if and only if there exists $\phi : P \to \mathrm{Diff}(F)$ satisfying $\mu'_g \phi_{pg} = \phi_p \mu_g$.

Problem Sheet 6

Associated Bundle

Exercise 6.1. Let $\pi : P \to B$ be a principal G-bundle and $E = P \times_\rho V$ an associated vector bundle. Fix a connection ω on P and let ∇ be the induced connection on E. Using the definition of ∇ show that

$$\nabla_X(fs) = f\nabla_X s + (\mathcal{L}_X f)\, s$$

for any $f \in C^\infty(B), s \in \Gamma(E)$ and $X \in \mathfrak{X}(B)$.

Exercise 6.2. Let $\pi : P \to B$ be a principal G-bundle where the Lie group G is abelian. Denote by $C^\infty(B, G)$ the group of smooth maps from B to G with multiplication given by pointwise multiplication and by $C^\infty(P, G)^G$ be the group $\{f : P \to G \mid f(pg) = g^{-1}f(p)g\}$. Show that the following map is a group isomorphism

$$\begin{aligned} C^\infty(B, G) &\to C^\infty(P, G)^G \\ \sigma &\mapsto f_\sigma = \sigma \circ \pi \end{aligned}$$

Exercise 6.3. Let $\pi : P \to B$ be a principal G-bundle and $E = P \times_G F$ be an associated fibre bundle. Recall that a map $f : M \to B$ defines a pullback bundle $f^*P \to M$, cf. Exercise 2.1.

(1) Show that the pullback principal G-bundle $\pi^*P \to P$ is trivial.

(2) Show that E is trivial if P is trivial.

(3) Show that the pullback bundle $\pi^*E \to P$ is always trivial.

Exercise 6.4. Let $\pi : P \to B$ be a principal G-bundle and $\omega \in \Omega^1(P, \mathfrak{g})$ a connection 1-form on P. Suppose that $\sigma \in \mathcal{G}$ is a bundle automorphism, i.e. a gauge transformation. Prove that $\sigma^*\omega$ is a connection 1-form on P which satisfies

$$\sigma^*\omega = \mathrm{Ad}\left(f^{-1}\right)\omega + f^*\theta$$

where f is the unique function such that $\sigma(p) = pf(p)$ and θ is the tautological 1-form on G.

Problem Sheet 7

Curvature and Connection 1-form

Exercise 7.1. Let E be a vector bundle with covariant derivative ∇. For two local trivializations differing by a gauge tranformation g prove that the two curvature matrices are related by $\Omega' = g\Omega g^{-1}$.

Exercise 7.2. Let E be a vector bundle with covariant derivative ∇ and F^∇ its curvature. Prove that

$$F^\nabla(X,Y)s = \nabla_X\nabla_Y s - \nabla_Y\nabla_X s - \nabla_{[X,Y]}s$$

[Hint: You may use the special case proved during the lecture.]

Exercise 7.3. Let $P \to B$ be a principal G-bundle and $\varphi : G \to H$ a homomorphism between Lie groups. Denote by P_φ the associated principal H-bundle. Show that for every connection 1-form $\omega \in \Omega^1(P, \mathfrak{g})$ there exists a unique connection 1-form $\omega' \in \Omega^1(P_\varphi, \mathfrak{h})$ such that

$$f^*\omega' = \varphi_* \circ \omega$$

where $f : P \to P_\varphi$ is defined by $f(p) = [(p, e)]$.
[Hint: Use Exercise 3.1.]

Exercise 7.4. Let G and $\pi : \widetilde{B} \to B$ be as in Exercise 5.2. Denote by $q : P_\rho \to B$ the principal G-bundle associated to the universal covering by $\rho : \pi_1(B) \to G$. Define the map $f : \widetilde{B} \to P_\rho$ by $f(p) = [(p, e)]$ as in Exercise 7.3.

(1) Use the previous exercise to show that there is a unique flat connection 1-form $\omega_\rho \in \Omega^1(P_\rho, \mathfrak{g})$ such that $f^*\omega_\rho = 0$.

(2) Let $q_P : P \to B$ be a principal G-bundle equipped with a connection 1-form ω_P and $f_1, f_2 : \widetilde{B} \to P$ two maps such that $q_P \circ f_i = \pi$ and $f_i^*\omega_P = 0$. Show that there exists a unique $g \in G$ such that $f_2 = f_1 g$.

(3) Let $q_P : P \to B$ be a principal G-bundle equipped with a connection 1-form ω_P and $f : \widetilde{B} \to P$ a map satisfying $q_P \circ f = \pi$ and $f^*\omega_P = 0$. Show that there exists a homomorphism $\rho_f : \pi_1(B) \to G$ with $f \circ \gamma = f\rho_f(\gamma)^{-1}$ for all $\gamma \in \pi_1(B)$ such that the map

$$\begin{aligned} \widetilde{B} \times G &\longrightarrow P \\ (p, g) &\mapsto f(p)g \end{aligned}$$

induces an isomorphism $\phi : P_{\rho_f} \to P$ with $\phi^*\omega_P = \omega_{\rho_f}$.

(4) Let $\rho_1, \rho_2 : \pi_1(B) \to G$ be two homomorphism and define $P_i = P_{\rho_i}$ and $\omega_i = \omega_{\rho_i}$ as before. Prove that there exists an isomorphism $\phi : P_1 \to P_2$ with $\phi^*\omega_2 = \omega_1$ if and only if there exists $g \in G$ such that $\rho_2 = g\rho_1 g^{-1}$.

Problem Sheet 8

Curvaure, Bianchi Identity and Principle Bundle Isomorphism

Exercise 8.1. Let P be a manifold and $\mathfrak{g}$ a Lie algebra. Recall that for $\omega \in \Omega^k(P, \mathfrak{g})$ and $\eta \in \Omega^l(P, \mathfrak{g})$ we defined $[\omega, \eta] \in \Omega^{k+l}(P, \mathfrak{g})$ by

$$[\omega, \eta]\left(X_1, \ldots, X_l\right) := \frac{1}{k!l!} \sum_{\sigma \in S_{k+1}} \operatorname{sgn}\left[\omega\left(X_{\sigma(1)}, \ldots, X_{\sigma(k)}, \eta\left(X_{\sigma(k+1)}, \ldots, X_{\sigma(n)}\right]\right.\right.$$

Prove the that this pairing has the following properties:

(1) $[\omega, \eta] = -(-1)^{kl}[\eta, \omega]$

(2) $[\eta, [\eta, \eta]] = 0$

(3) $d[\omega, \eta] = [d\omega, \eta] + (-1)^k[\omega, d\eta]$

Exercise 8.2. Let P be a principal G-bundle and $\mathfrak{g}$ be the Lie algebra of G. If $\omega \in \Omega^1(P, \mathfrak{g})$ is a connection 1-form with curvature Ω, prove the Bianchi-identity

$$d\Omega = [\Omega, \omega]$$

Deduce from this the form of the Bianchi identity proved in the lectures:

$$d\Omega\Big|_{\ker \omega} \equiv 0$$

Exercise 8.3. In the setting of the previous exercise, Ω corresponds to some $F \in \Omega^2(B, \mathrm{Ad}(P))$, where $\mathrm{Ad}(P) = P \times_{\mathrm{Ad}} \mathfrak{g}$. The form ω induces a covariant derivative ∇ on $\Gamma(\mathrm{Ad}(P))$, which is extended to $\overline{\nabla}$ on $\Omega^k(B, \mathrm{Ad}(P))$. Prove

$$\overline{\nabla} F = 0$$

Problem Sheet 9

Euler Class on S^2 and Curvature Integrals

Exercise 9.1. Consider $\mathbb{R}^3$ with the standard scalar product $\langle\cdot,\cdot\rangle_0$ and compatible flat covariant derivative ∇^0 on $T\mathbb{R}^3$.

For the unit sphere $S^2 \subset \mathbb{R}^3$, we consider

$$TS^2 \subset T\mathbb{R}^3\Big|_{S^2} = TS^2 \oplus \mathbb{R}$$

with the trivial summand spanned by the outward unit normal N to $S^2 \subset \mathbb{R}^3$. On $TS^2 \to S^2$, we define a covariant derivative ∇ by

$$\nabla_X Y = \operatorname{pr}\left(\nabla^0_X Y\right)$$

where $\operatorname{pr} : T\mathbb{R}^3\Big|_{S^2} \to TS^2$ is the projection with kernel spanned by N.

(1) Check that ∇ is compatible with the metric $\langle\cdot,\cdot\rangle$ on TS^2 given by the restriction of $\langle\cdot,\cdot\rangle_0$.

(2) Compute an explicit representative for $e\left(TS^2\right) \in H^2_{dR}\left(S^2\right)$ from the curvature of ∇.

(3) Prove that $\displaystyle\int_{S^2} e\left(TS^2\right) = 2$.

[Hint: $f : [0,2\pi] \times [0,\pi] \to S^2, (u,v) \mapsto (\cos u \cdot \sin v, \sin u \cdot \sin v, \cos v)$ gives a parametrization of S^2. Normalize $\dfrac{\partial f}{\partial u}, \dfrac{\partial f}{\partial v}$ to unit length and check that this gives an orthonormal frame for TS^2. The calculation in (2) and (3) is easy using this frame.]

Exercise 9.2. Let $P \to B$ be a principal SO(2)-bundle. Give a definition of an Euler class $e(P) \in H^2(B,\mathbb{R})$ which does not use the associated vector bundle, but instead a connection 1-form and local expression for the curvature of this form. Prove that it is independent of the choices made. Then show that the class you defined coincides with the Euler class of the associated vector bundle. Use this and Exercise 4.1 to compute $\displaystyle\int_{\mathbb{CP}^1} e$, where e is the Euler class of the Hopf bundle.

Exercise 9.3. Show that the Euler class is functorial under pullbacks, i.e. given a smooth map $f : N \to M$ and an oriented rank 2 bundle E over M, one has:

$$e\left(f^* E\right) = f^* e(E)$$

Exercise 9.4. Let B be a manifold and let E be an oriented vector bundle with a decomposition $E = L \oplus \underline{\mathbb{R}}$, where L is a line bundle and $\underline{\mathbb{R}}$ the trivial line bundle. Show that $e(E) = 0$. Deduce that for an arbitrary E (still oriented, of rank two) the Euler class $e(E)$ vanishes if there exists a nowhere vanishing section $s : B \to E$.

Problem Sheet 10

Hodge Star Operator and Yang-Mills Theory

Exercise 10.1. Let (M, g) be an n-dimensional oriented Riemannian manifold and $*$ the Hodge star operator.

(1) Prove that
$$** : \Omega^k(M) \longrightarrow \Omega^k(M)$$
is given by
$$** = (-1)^{k(n-k)}$$

(2) Determine the even dimensions $n = 2k$ where $** = 1$ on $\Omega^k(M)$. In these dimensions we can define self-dual and anti-self-dual k-forms ω, satisfying $*\omega = \omega$ and $*\omega = -\omega$, respectively.

Exercise 10.2. Let (M, g) be a closed (compact without boundary) n-dimensional oriented Riemannian manifold. The Laplace operator on k-forms is defined by

$$\Delta = dd^* + d^*d : \Omega^k(M) \to \Omega^k(M)$$

where d^* is the formal adjoint of d. A form ω is called harmonic if $\Delta\omega = 0$. Prove that

$$\omega \text{ is harmonic } \iff d\omega = 0 = d^*\omega \iff *\omega \text{ is harmonic.}$$

Exercise 10.3. Let (M, g) be a Riemannian 4-manifold with principal bundle $P \to M$. Prove that the Yang-Mills functional is invariant under conformal change of the metric, i.e. when replacing g by g' with

$$g' = e^{2\lambda} g$$

where $\lambda \in C^\infty(M)$ is an arbitrary smooth function on M.

Exercise 10.4.

(1) Prove that the connection A on the Hopf bundle $S^3 \to S^2$, introduced in Exercise 4.3, satisfies the Yang-Mills equation if S^2 has the standard round Riemannian metric.

(2) Prove that the Yang-Mills moduli space for the Hopf bundle $S^3 \to S^2$ over the round sphere S^2 consists of a single point.

Problem Sheet 11

Yang-Mills Theory

Exercise 11.1. Let ∇ be a covariant derivative on $E \to B$ and $\overline{\nabla}$ its extension on $\operatorname{End} E$ defined by

$$\left(\overline{\nabla}_X \varphi\right) s = \nabla_X(\varphi(s)) - \varphi\left(\nabla_X s\right)$$

for all $X \in TB, s \in \Gamma(E), \varphi \in \Gamma(\operatorname{End} E)$.

(1) Prove that $\overline{\nabla}$ is indeed a covariant derivative on $\operatorname{End} E$.

(2) Prove that

$$F^{\overline{\nabla}}(X,Y)\varphi = \left[F^{\nabla}(X,Y), \varphi\right]$$

where the right-hand side is the commutator of endomorphism

$$[\psi, \varphi] = \psi \circ \varphi - \varphi \circ \psi$$

Exercise 11.2. Let $P \to B$ be a principal G-bundle.

(1) Prove that if P admits a reduction to $S^1 \subset G$, then P admits a Yang-Mills connection for any Riemannian metric on B.

(2) If B is 4-dimensional, is the same statement true for self-dual or anti-self-dual Yang-Mills connections?

Exercise 11.3. Let $P \to B$ be a principal G-bundle with gauge group $\mathcal{G}$ and space of connections $\mathcal{C}$. Determine all possible stabilizers $\operatorname{Stab}(\omega) \subset \mathcal{G}$ for the $\mathcal{G}$-action on $\mathcal{C}$ in the cases $G = \mathrm{SU}(2)$ and $G = \mathrm{SO}(3)$.

Exercise 11.4. Consider the principal SU(2)-bundle $S^7 \to S^4$ defined in an analogous way to the Hopf bundle $S^3 \to S^2$ when replacing the complex number with quaternions.

(1) In analogy with Exercise 4.3 define a connection 1-form $A \in \Omega^1\left(S^7, \mathfrak{su}(2)\right)$.

(2) Prove that A satisfies the Yang-Mills equation for the standard round Riemannian metric on S^4.

Problem Sheet 12

4-manifolds

Exercise 12.1. Let V be an oriented 4-dimensional $\mathbb{R}$-vector space with a scalar product.

(1) Given an oriented orthonormal basis $\alpha_1, \ldots, \alpha_4 \in V$, write out explicit orthonormal bases for $\Lambda^2_+ V$ and $\Lambda^2_- V$, derived from the α_i.

(2) Given an arbitrary vector space W and an $\alpha \neq 0 \in V$, show the linear map

$$\begin{aligned} \pi_- \circ \alpha \otimes : V \otimes W &\to (\Lambda_- V) \otimes W \\ \beta \otimes w &\mapsto (\alpha \wedge \beta)_- \otimes w \end{aligned}$$

has kernel consisting of elements of the form $\alpha \otimes w \in V \otimes W$.

(3) Show that the map in (2) is surjective.

Remark. This completes the proof that the symbol sequence of the twisted half de Rham complex of an oriented Riemannian 4-manifold is exact.

Exercise 12.2. Let $Q : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{Z}$ be a positive definite symmetric bilinear form. Assume Q is unimodular in the sense that $\det Q = \pm 1$. Let m be one half the number of solutions to the equation

$$Q(\alpha, \alpha) = 1$$

(1) Prove that $m \leqslant r$, with equality if, and only if, Q can be diagonalized over $\mathbb{Z}$.

(2) Prove that the symmetric bilinear form corresponding to the quadratic form

$$Q_{E_8}(x_1, \ldots, x_8) := 2 \sum_{i=1,\ldots,8} x_i^2 - 2 \sum_{i=1,\ldots,6} x_i x_{i+1} - 2x_5 x_8$$

is positive definite and unimodular, but not diagonalizable over $\mathbb{Z}$.

Exercise 12.3. For a closed oriented 4-manifold X, denote by Q_X its intersection form.

(1) Compute $Q_{S^2 \times S^2}$ and $Q_{\mathbb{CP}^2}$.

(2) Let $P(x, y) := x^2 - y^2$. Show that $Q_{S^2 \times S^2}$ is equivalent to P over the reals, but not over the integers.

Remark. One may see check $P = Q_{\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}}$. So even though $S^2 \times S^2$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ have the same Betti-numbers, they are distinguished by their intersection form.