- gauge theory: geometry of fibre bundles / esp- principal bundles
- gauge field: connection
- field strength: curvature
- choice of gauge: local trivialization
- local gauge transformation: change of trivialisation
- global gauge tranformation: bundle automoprhisms

principal G-bundles. G is a Lie-Group

1 Lie Groups and Lie algebras

Definition 1.1. A **Lie Group** is a smooth mfd. which is also a group in such way that

$$m : G \times G \to G \qquad\qquad i : G \to G$$
$$(a,b) \mapsto a \cdot b \qquad\qquad a \mapsto a^{-1}$$

are smooth maps.

Remark 1.2. i is a diffeomorphism $g \in G$ defines $l_g : G \to G, a \mapsto g \cdot a$. This is smooth since m is. l_g is a diffeomorphism with inverse $l_{g^{-1}} = (l_g)^{-1}$

Definition 1.3. A **Lie algebra** (/ is a vector space V with a map

$$[,] : V \times V \to V$$

s.t.:

(1) is bilinear

(2) is skew- symmetric

(3) satisfies the Jacobi- identity:

$$[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$$

Example 1.4. *M is a smooth mfd.* $V = \mathfrak{X}(M) = \Gamma(TM)$ *the vector space of smooth vector fields on M. V is a Lie algebra w.r. to [,] defined by:*

$$L_{[X;Y]}f = L_X L_Y f - L_Y L_X f \quad \forall X,Y \in X(M), f \in C^\infty(M)$$

Take $M = G$ a Lie Group, Let $\mathfrak{g} \subset \mathfrak{X}(G)$ be the subspace of **left-invariant vector fields**, that is vector fields X which satisfy:

$$D_a l_g(X(a)) = X(g \cdot a) \quad \forall a, g \in G$$

$\mathfrak{g}$ is closed under $[,]$ (exercise!)

Definition 1.5. $\mathfrak{g} = L(G)$ is the **Lie algebra of G**.

Example 1.6.
- *G any group, with the discrete topology. If G is countable, then this is a 0-dim. mfd and m,i are smooth, so G is a Lie group. In this case g is trivial.*
- $G = \mathbb{R}^n, m = +$. *$(G,+)$ is a Lie group,* $n = dimG$. $g = \mathbb{R}^n, [,] \equiv 0$
- $G = S^1 = \{z \in \mathbb{C} | |z| = 1\}$ *is a 1-dim Lie-Group with* $g = \mathbb{R}, [,] \equiv 0$.

- $G = S^3 = \{q \in \mathbb{H} || q| = 1\}$ *with m given by quaternian multiplication.* $g = \mathbb{R}^3, [,] \not\equiv 0$

  Consider

$$ev : \mathfrak{g} \to T_e G$$
$$X \mapsto X(e)$$

Proposition 1.7. *ev is an isomorphism of vector spaces.*

Proof. ev is linear. suppose $ev(x) = 0$. Then $X(e) = 0$ ad $X(g) = D_e l_g(X(e)) = 0 \forall g \in G$, so $X \equiv 0$.So ev is injective.
Let $v \in T_l G$. Define $X \in \mathfrak{g}$ by $X(g) = D_e l_g(v)$.

$$D_g l_{g'}(X(g)) = D_g l_{g'}(D_e l_g(v)) = (D_e l_{g'g})(v) = X(g'g)$$

so X is indeed left invariant. Then $ev(X) = v$, so ev is surjective. □

Corollary 1.8. *Every Lie group G has a well-defined dimension as a mfd. (i.e. all connected components have the same dimension) and this coincides with the dimension of* $\mathfrak{g}$ *as a vector space.*

Corollary 1.9. *Every Lie group is **parallelizable**, i.e. it has trivial tangent bundle.*

Proof. Consider $G \times \mathfrak{g} \to TG, (g, X) \mapsto X(g)$.
Claim: "this is an ismorphism of vector bundles over G"

Proof. the map is smooth. Restricted to $\{g\} \times \mathfrak{g}$ we have $\mathfrak{g} \to T_g G, X \mapsto X(g)$. This is a linear injective map between vector spaces of same dimension, so it is an isomorphism. □

□

Definition 1.10. A **1-parameter subgroup** of a Lie group G is a smooth map $s : \mathbb{R} \to G$ with $s(0) = e$ and $s(t_1 + t_2) = s(t_1)s(t_2) \forall t_1, t_2 \in \mathbb{R}$.

$$(\mathbb{R}, +) \underset{s}{\to} (G, .)$$

is a homomorphism which is smooth.

Proposition 1.11. *Let G be a Lie Group with Lie algebra* $\mathfrak{g}$. *Fix* $X \in \mathfrak{g}$. *Then:*

1. *X is complete*
2. *there is a unique 1- parameter subgroup* $s_x : \mathbb{R} \to G$ *s.t.* $\dot{s}_x(0) = X(e)$ *The flow of X is given by* $\varphi_t(g) = g s_x(t)$

Proof. 1 there exists a local flow for X around e: there is a $\epsilon > 0$ and U an open neighbourhood of e in G s.t.: $\varphi : (-\epsilon, \epsilon) \times U \to G, (t, g) \mapsto \varphi_t(g)$. is defined ad smooth with $\dot{\varphi_t}(g) = X(g)$. Take $a \in G$ arbitrary. Consider

$$\phi : (-\epsilon, \epsilon) \times l_a(U) \to G, (t, b) \mapsto l_a\varphi(_t(l_{a^{-1}}(b)) = a \cdot \varphi_t(a^{-1}b)$$

This is a smooth local flow for X around a. Since ϵ does not depend on a, these local flows are part of a global flow for X.

2 suppose we have s_x so that $\varphi_t(g) = g \cdot s_x(t)$ for $g \in U, |t| < \epsilon$. Then $\phi_t(a) = a \cdot \varphi_t(e) = a \cdot e \cdot s_x(t)$ By (1) X has a global flow $\varphi : \mathbb{R} \times G \to G$. Define $s_x(t) = \varphi_t(e) = \varphi(t, e))$. Them $s_x(0) = e$ and $s_x(t_1 + t_2) = \varphi_{t_1+t_2}(e) = \varphi_{t_1} \circ \varphi_{t_2}(e) = s_x(t_1) \cdot s_x(t_2)$. This means that s_x is a 1-parameter subgroup. Claim: "The global flow of X is given by $\varphi_t(g) = gs_x(t)$"

□

<u>*Vorlesung 2:*</u>

Example 1.12.

$$G = GL_n(\mathbb{R}) = \{Mat(n \times n, \mathbb{R}) | det \neq 0\} \subset \mathbb{R}^{n^2}$$

is a Lie group $x \in \mathfrak{g} = Mat(n \times n, \mathbb{R}) \Rightarrow s_x(t) = exp(tx)$

Definition 1.13. For any Lie Group G with algebra $\mathfrak{g}$ define

$$exp : \mathfrak{g} \to G \quad x \mapsto s_x(1)$$

the **exponential map of G**.

Lemma 1.14. $exp(tx) = s_x(t)$

Proposition 1.15. *The map exp is smooth.*

Definition 1.16. A **homomorphism** $\varphi : G_1 \to G_2$ **of Lie Groups** is a smooth map which is also a group homomorphism. $\varphi(e) = e$

$$\begin{array}{ccc} T_eG_1 & \xrightarrow{D_e\varphi} & T_eG_2 \\ {\scriptstyle ev}\uparrow & & {\scriptstyle ev}\uparrow \\ \mathfrak{g}_1 & \xrightarrow{\varphi_*} & \mathfrak{g}_2 \end{array}$$

$\varphi_*$is the unique map which makes the diagram commute.

Proposition 1.17. φ_* *is a Lie algebra homomorphism, i.e. a linear map satisfying* $\varphi_*[X, Y] = [\varphi_*X, \varphi_*Y]$.

Proof. φ_* is linear bec. it is a composition of linear maps. $\square$

Lemma 1.18. *If $\varphi : G_1 \to G_2$ is a Lie Group homomorphism, then $(D_g\varphi)(X(g)) = (\varphi_* X)(\varphi(g))$ for all $g \in G_1, X \in \mathfrak{g}$*

Proof.

$$
\begin{aligned}
(D_e\varphi(X(g)) &= (D_g\varphi)(D_e\varphi_g(X(e))) = D_g\varphi(D_e\varphi_g(ev(X)) \\
(\varphi_* X)(\varphi(g)) &= (ev^{-1} \circ D_e\varphi \circ ev)(X)(\varphi(g)) = (ev^{-1}(\underbrace{D_e\varphi(X(e))}_{\in T_eG_2})(\varphi(g)) \\
&= (D_e\varphi_{\varphi(g)})(D_e\varphi(X(e))) \underset{(1)}{=} D_e\varphi_{\varphi(g)} \circ D_e\varphi(ev(X))
\end{aligned}
$$

(1):$\varphi(l_g(a)) = \varphi(g \cdot a) = \varphi(g) \cdot \varphi(a) = l_{\varphi(g)}(\varphi(a))$, $\varphi \circ l_g = l_{\varphi(g) \circ \varphi}$ since it is diff. at $e \in G_1$ we follow: $D_g\varphi \circ D_e l_g = D_e l_{\varphi(g)} \circ D_e\varphi$

$$
\begin{array}{ccc}
G_1 & \xrightarrow{\varphi} & G_2 \\
\downarrow{\scriptstyle l_g} & & \downarrow{\scriptstyle l_{\varphi(g)}} \\
G_2 & \xrightarrow{\varphi} & G_2
\end{array}
$$

$\square$

Lemma 1.19. *Let $\varphi : M \to N$ be a smooth map of smooth mfds. Suppose $X, Y \in \mathfrak{X}(M)$ and there exist vector fields $\bar{X}, \bar{Y} \in \mathfrak{X}(M)$ s.t.:*

$$D_x\varphi(X(x)) = \bar{X}(\varphi(x)) \quad D_x\varphi(Y(x)) = \bar{Y}(\varphi(x)) \quad \forall x \in M$$

Then

$$D_x\varphi[X, Y](x) = [X, Y](x)$$

Proof. Let $f \in C^\infty(N)$. Then:

$$
\begin{aligned}
(L_{D_x\varphi(x)} f)(\varphi(x)) &= L_X f \circ \varphi)(x) L_{\bar{X}} f(\varphi(x)) \\
L_{D\varphi[X,Y]} f &= L_{[X,Y]}(f \circ \varphi) = L_X L_Y - L_Y L_X(f \circ \varphi) \\
&= L_X(L_Y(f \circ \varphi)) - L_Y(L_X(f \circ \varphi)) = L_X(L_{\bar{Y}} f) - L_Y(L_{\bar{X}} f)) \\
&= L_{\bar{X}} L_{\bar{Y}} f - L_{\bar{Y}} L_{\bar{X}} f = L_{[\bar{X}, \bar{Y}]} f
\end{aligned}
$$

$\square$

now proof of 1.17:

Proof. Set $\bar{X} = \varphi_* X, \bar{Y} = \varphi_* Y$. By 1.18 this means that 1.19 is applicable. So 1.19 gives us $\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y]$ $\square$

Recall the Frobenius Thm:

Theorem 1.20. *Let M be a smooth mfd., dim M=n and $E \subset TM$ a smooth subbundle of rank $0 \leq k \leq n$. The following conbditions on E are equivalent:*

(1) E is integrable (i.e. E=TF, F a k-dim foliation on M. For every $x \in M$ there is a local k-dim. submanifold trough x s.t. the tangents to the submanifold are exactly E i.e. $T_yS = E_y \forall y \in S$)

(2) E is involutive (i.e. $\Gamma(E) \subset \mathfrak{X}(M)$ is closed under [].)

(3) around every point $x \in M$ there is a chart $(U, \varphi), \varphi : U \to \varphi(U) \subset \mathbb{R}^n$ such that $E|_U$ is the span of $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_k}$

Vorlesung 3: G a Lie group

Definition 1.21. A subset $H \subset G$ is a **Lie subgroup** if it is an abstract subgroup and it can be given a smooth mfd. structure such that it is a Lie group and the inclusion $i : H \hookrightarrow G$ is an immersion.

Example 1.22. $G = T^2 = \mathbb{R}^2/\mathbb{Z}^2 = S^1 \times S^1$ *Every line trough e gives a 1-par. subgroup. If the slope of the line is in $\mathbb{Q}$. Then the 1-par. subgroup is an embedded submfd. diffeomorphis to S^1. If the slope is irrational, then the 1-par. subgroup is an inj. immersed copy of $\mathbb{R}$ with dense image in T^2. In both cases, these are Lie subgroups.*

Theorem 1.23. *These is a bijective correspondence between connected Lie subgroups of G and Lie subalgebras of $\mathfrak{g}$.*

Proof. First direction:
Suppose $H \subset G$ is a Lie subgroup with incl. $i : H \hookrightarrow G$. Then i is an immersion and so $D_h i$ is injective $\forall h \in H$.

$$\begin{array}{ccc} T_eH & \xrightarrow{D_e i} & T_eG \\ {\scriptstyle ev}\uparrow & & \uparrow{\scriptstyle ev} \\ \mathfrak{h} & \xrightarrow{i_*} & \mathfrak{g} \end{array}$$

i_* is injective so we may identify $\mathfrak{h}$ with $i_*(\mathfrak{h}) \subset \mathfrak{g}$. This $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra corresponding to $H \subset G$. *Second direction:*
Suppose $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{ev} & T_eG \\ \downarrow{\scriptstyle \subset} & & \downarrow{\scriptstyle \subset} \\ \mathfrak{h} & \xrightarrow{ev|_{\mathfrak{g}}} & E_e \end{array}$$

Define $E_g = D_e l_g(E_e)$. This is the evaluation of h at g for all $g \in G$.
Claim: "This defines a smooth subbundle $E \subset TG$"

Proof. $k = dim\mathfrak{h}, X_1, ..., X_k$ a basis for $\mathfrak{h}$. These are smooth v.f.s. and they are everywhere linearly independent. $E = span\{X_1, ..., X_k\} \rightsquigarrow E$ is a smooth subbundle. □

Claim: "E is involutive"

Proof. $\mathfrak{h} \subset \mathfrak{g}$ Lie subalgebra $\Rightarrow [X_i, X_j] \in \mathfrak{h}$. Every $X, Y \in \Gamma(E)$ are of the form $X = \sum_{i=1}^{k} x_i X_i$ and $Y = \sum_{i=1}^{k} y_i Y_i$.

$$\Rightarrow [X, Y] \text{ is a linear comb. of } [X_i, Y_j] \text{ ad } X_i \Rightarrow [X, Y] \in \Gamma(E)$$

By 1.20 E is integrable. Let $\mathfrak{F}$ be the foliation whose leaves L_g are the maximal connected integral mfds. for E. Let $H = L_e$ □

Claim: "H is a Lie subgroup with Lie algebra $\mathfrak{h}$"

Proof. H has the smooth mfd. str. of a leaf of $\mathfrak{F}$, ad is inj. immersed. For any $g \in G$ we have $L_g = l_g(L_e)$. $D_p l_g(E_p) = E_{gp}$, so E is invariant under l_g.$\Rightarrow l_g$ maps leaves of $\mathfrak{F}$ to leaves of $\mathfrak{F}$, so $l_g(L_e)$ is a leaf of $\mathfrak{F}$ through the point g. by uniqueness of leaves, $l_g(L_e) = L_g$.
Claim "$g \in G$ is in H if and only if $L_g = H$"

Proof. If $g \in H$, then $L_g = H$ by uniqueness of leaves. Conversely, assume $L_g = H$. Then $H = L_e = l_{g^{-1}}(L_g) = g^{-1} \cdot H \Rightarrow g \in H$ □

Suppose $h_1, h_2 \in H = L_e$. Then $h_1 \cdot L_e = L_{h_1} = H \Rightarrow h_1, h_2 \in H$. if $h \in H$, then

$$h^{-1} \cdot L_e = L_{h^{-1}} = L_e = H \Rightarrow h^{-1} \in H$$

$\Rightarrow H \subset G$ is an abstract subgroup. Multiplication and inversion are smooth, so H is a connected Lie subgroup. The Lie algebra of H is the $\mathfrak{h}$ we started with.

$$\mathfrak{h} \rightsquigarrow H \rightsquigarrow \mathfrak{h} \quad (1)$$
$$H \rightsquigarrow \mathfrak{h} \rightsquigarrow H \quad (2)$$

□

□

Example 1.24.
- $G = T^2, \mathbb{R} \subset \mathfrak{g} = \mathbb{R}^2$
- *In $GL_n(\mathbb{R})$ we can consider subgroups $O(n), SO(n), U(n/2)$ for n even,...*

2 Smooth Lie group actions

M a smooth mfd. G a Lie group.

Definition 2.1. A **smooth action** on the left **of G on M** is a smooth map:

$$\varphi : G \times M \to M \quad (g, p) \mapsto \varphi_g(p) = g(p) = g \cdot p$$

s.t.:

$$\varphi_e = Id_M \quad \varphi_{g_1}(\varphi_{g_2}(p)) = \varphi_{g_1 g_2}(p) \quad \forall g_1, g_2 \in G p \in M$$

smooth action on the right is a smooth map :

$$\varphi : M \times G \to M \quad (p, g) \mapsto \varphi_g(p) = g(p) = p \cdot g$$

s.t.:

$$\varphi_e = Id_M \quad \varphi_{g_1}(\varphi_{g_2}(p)) = \varphi_{g_2 g_1}(p) \quad \forall g_1, g_2 \in G p \in M$$

Left and right actions agree if and only if G is Abelian.

Example 2.2. $G = (\mathbb{R}, +)$. *In this case any G-action is a smooth flow on M.*

Terminology:
Assume G acts smoothly on M. The **orbit of a point** $p \in M$ is the subset $\{\varphi_q(p) | g \in G\} \subset M$.

$\varphi^p : G \to M \quad g \mapsto \varphi_g(p)$ is a smooth map for every p, parametrizing the orbit of p

For every $p \in M$ we define $G_p := \{g \in G | \varphi_g(p) = p\}$, the **isotropy group** of p. This is a Lie subgroup. It is also closed as a subset of G. The action is **effective** if :

$$\forall g \in G/e \exists p \in M \text{s.t.:} \varphi_g(p) \neq p$$

The action is **transitive** if :

$$\forall p, q \in M \exists g \in G \text{s.t.:} \varphi_g(p) = q$$

Example 2.3.

- $G \times g \to G \quad (g, p) \mapsto g \cdot p = l_g(p)$ *is a transitive effective left action.*
- *similarly:*
- $G \times g \to G \quad (g, p) \mapsto p \cdot g = r_g(p)$ *is a transitive effective right action.*
- $$\varphi : G \times G \to G \quad (g, p) \mapsto \varphi_g(p) = gpg^{-1}$$

  $\varphi_{g_1} \varphi_{g_2}(p) = \varphi_{g_1}(g_2 p g_2^{-1}) = g_1((g_2 p g_2^{-1}) g_1^{-1} = (g_1 g_2) p (g_1 g_2)^{-1} = \varphi_{g_1 g_2}(p)$
  $\rightsquigarrow$ *This is a left action of G on itself!*

<u>*Vorlesung 4:*</u>

Proposition 2.4. *If $\varphi : G \to H$ is a Lie group homomorphism, then the following diagramm commmutes:*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\varphi_*} & \mathfrak{h} \\ \Big\downarrow{\scriptstyle exp} & & \Big\downarrow{\scriptstyle exp} \\ G & \xrightarrow{\varphi} & H \end{array}$$

Proof. Since φ is a homomorphism the upper diagramm commutes:

$$\begin{array}{ccc} T_eG & \xrightarrow{D_e\varphi} & T_eH \\ {\scriptstyle ev}\Big\uparrow & & {\scriptstyle ev}\Big\uparrow \\ \mathfrak{g} & \xrightarrow{\varphi_*} & \mathfrak{h} \\ \Big\downarrow{\scriptstyle exp} & & \Big\downarrow{\scriptstyle exp} \\ G & \xrightarrow{\varphi} & H \end{array}$$

Take $X \in \mathfrak{g}$. Then:

$$exp(\varphi_*(tX)) = exp(t\,\underbrace{ev^{-1} \circ D_e\varphi \circ ev(X)}_{=:Y}) = exp(tY)$$

$\varphi(exp(tX))$ is a 1-par.subgroup of H. The conclusion follows if we check that these two 1-par. subgroups are subgroups for the same element in $\mathfrak{h}$.

$$\frac{d}{dt}\varphi(exp(tX))|_{t=0} = D_e\varphi(\frac{d}{dt}exp(tX)|_{t=0}) = D_e\varphi(ev(X))$$

By uniqueness of 1-par. subgroup follows equivalence. □

Let $\varphi : G \times M \to M$ be a smooth action of G on M. Let G_p be the isotropy subgroup of a point $p \in M$.

Lemma 2.5. *φ induces a linear representation $\zeta_p : G_p \to GL(T_pM)$. This is called the **isotropy representation** at p.*

Proof. Let $g \in G_p \subset G$, so $\varphi_g(p) = p \Rightarrow D_p\varphi_g : T_pM \to T_pM$. We define

$$\zeta_p : G_p \to GL(T_pM), g \mapsto D_p\varphi_g$$

φ_g diffeo. $\Rightarrow D_p\varphi_g$ invertible for all $p \in M$. If $g_1, g_2 \in G_p$, then $D_p\varphi_{g_1g_2} = D_p(\varphi_{g_1} \circ \varphi_{g_2} = D_{g_2(p)}\varphi_{g_1} \circ D_p\varphi_{g_2} = D - pg_1 \circ D_pg_2$, so ζ_p is a homomorphism. □

Consider

$$a : G \times G \to G \quad (g, p) \mapsto gpg^{-1}$$

This is a left action of G on G. In this action $G_e = G$. This gives isotropy rep.:

$$\zeta_e : G \to GL(T_eG) \cong Gl(\mathfrak{g})$$

Definition 2.6. The **adjoint representation** of G on $\mathfrak{g}$ is:

$$Ad = \zeta_e : G \to GL(\mathfrak{g})$$

$$\begin{array}{ccc} T_eG & \xrightarrow{D_eAd} & T_eH \\ ev \uparrow & & ev \uparrow \\ \mathfrak{g} & \xrightarrow{Ad_* := ad} & End(\mathfrak{g}) \\ \downarrow exp & & \downarrow exp \\ G & \xrightarrow{Ad} & GL(\mathfrak{g}) \end{array}$$

Definition 2.7.

$$ad : \mathfrak{g} \to End(\mathfrak{g})$$

is Ad_*.

Definition 2.8. The **Center of G**:

$$C(G) := \{g \in G | gg` = g`g \quad \forall g` \in G\}$$

If $g \in C(G)$, then $a_g = Id_G$.

Lemma 2.9. *$C(G) \subset Ker(Ad)$. In particular Ad is the trivial rep. if G is abelian.*

Proof.

$$a_g : G \to G \quad p \mapsto gpg^{-1}$$

is a Lie group homomorphism. $a_g(p \cdot g) = g(p \cdot q)g^{-1} = (gpg^{-1})(gqg^{-1}) = a_g(p) \cdot a_g(q)$.

$$\begin{array}{ccc} T_eG & \xrightarrow{D_ea_g} & T_eG \\ ev \uparrow & & ev \uparrow \\ \mathfrak{g} & \xrightarrow{(a_g)_*} & \mathfrak{g} \\ \downarrow exp & & \downarrow exp \\ G & \xrightarrow{a_g} & G \end{array}$$

Claim:"$(a_g)_* = Ad(g)$"

Proof. $Ad(g)$ was defined as $D_e a_g$. But so was $(a_g)_*$. $\square$

Take V a finite dim. $\mathbb{R}$-vector space and $G = GL(V)$.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{Ad(g)} & \mathfrak{g} \\ \downarrow{\scriptstyle exp} & & \downarrow{\scriptstyle exp} \\ G & \xrightarrow{a_g} & G \end{array} \qquad \begin{array}{ccc} GL(v) & \xrightarrow{Ad} & GL(End(V)) \\ {\scriptstyle exp}\uparrow & & {\scriptstyle exp}\uparrow \\ End(V) & \xrightarrow{ad} & End(End(V)) \end{array} \qquad \begin{array}{ccc} GL(V) & \xrightarrow{a_g} & GL(V) \\ {\scriptstyle exp}\uparrow & & {\scriptstyle exp}\uparrow \\ End(V) & \xrightarrow{Ad(g)} & End(V) \end{array}$$

Claim: "$Ad(g)(M) = gMg^{-1}$"

Proof.

$$\begin{aligned} Ad(g)(M) &= \frac{d}{dt} a_g(exp(tM))|_{t=0} = \frac{d}{dt}(gexp(tM)g^{-1})|_{t=0} \\ &= g(\frac{d}{dt} exp(tM)|_{t=0})g^{-1} = gMg^{-1} \end{aligned}$$

$\square$

$\square$

Proposition 2.10. *Consider $G = GL(V)$ and $ad : \mathfrak{g} \to End(\mathfrak{g})$. Then $ad(X)(Y) = [X, Y]$.*

Proof.

$$\begin{aligned} ad(X)(Y) &= \frac{d}{dt} Ad(exp(tY))(Y)|_{t=0} = ev^{-1}(\frac{d}{dt} D_e a_{exp(tX)}(Y_e)|_{t=0}) \\ &= ev^{-1}(\frac{d}{dt} D_{exp(tX)} r_{exp(-tx)} \circ D_e l_{exp(tX)}(Y_e)|_{t=0}) \\ &= ev^{-1}(\frac{d}{dt} D_{exp(tX)} r_{exp(-tX)}(Y_{exp(tX)})|_{t=0}) \\ &= ev^{-1}((L_X Y)_e) = ev^{-1}([X, Y](e)) = [X, Y] \end{aligned}$$

$\square$

<u>*Vorlesung 5:*</u> G a Lie group, $g \in G$:

$$a_g : G \to G, h \mapsto ghg^{-1} \quad Ad_g = D_e a_g : T_e G \mapsto T_e G$$

The adjoint representation of G is the isotropy representation of a_g at $e \in G$ i.e.:

$$Ad : G \to GL(\mathfrak{g}) \quad g \mapsto Ad_g$$

Lemma 2.11. *Let G be connected. Then:* $ker(Ad) = C(G)$ *it the centre of G.*

Proof. If $g \in C(G)$, Then $a_g = Id$ so $Ad_g = D_e a_g = D_e Id = Id$. This means $g \in ker(Ad)$. Conversly , if $g \in ker(Ad)$:

$$\begin{array}{ccc} G & \xrightarrow{a_g} & G \\ {\scriptstyle exp}\uparrow & & {\scriptstyle exp}\uparrow \\ \mathfrak{g} & \xrightarrow{Ad_g} & \mathfrak{g} \end{array}$$

Let $X \in \mathfrak{g}$, then:

$$gexp(X)g^{-1} = a_g(exp(X)) = exp(Ad_g(X)) = exp(X)$$

In particular, since exp is a diffeomorphism near $0 \in \mathfrak{g}$, $\mathfrak{g}$ commutes with an open neighborhood of $e \in G$. Since any open nbhd. of $e \in G$ .... the connected component of G containing e (exercise), and since G is connected, g commutes with al of G, i.e. $g \in C(G)$.

□

Definition 2.12. The **centre of** $\mathfrak{g}$ is the Lie algerba:

$$C(\mathfrak{g}) := \{X \in \mathfrak{g} | [X, Y] = 0 \forall Y \in \mathfrak{g}\}$$

Corollary 2.13. *Let G be a connected Lie group. The* $C(G)$ *is a closed Lie subgroup of G, where lie algebra is the centre of* $\mathfrak{g}$.

Proof. By previous Lemma $C(G) = ker(Ad : G \to GL(\mathfrak{g}))$,i.e. $C(G)$ is a closed subgroup of G. By the "Carton closed subgroup thm.", $C(G)$ is an embedded Lie subgroup of G.
$\underline{C(\mathfrak{g}) \subset LC(G)}$:
$X \in C(\mathfrak{g})$, Then for all $t \in \mathbb{R}$ we have:

$$Ad_{exp(tX)} = exp(\underbrace{ad_X}_{=0 \text{ since } X \in C(\mathfrak{g}}) = id$$

i.e.: $exp(tX) \in ker(Ad) = C(G)$. Hence, $X \in LG(G)$.
$\underline{C(\mathfrak{g}) \supset LC(G)}$:
$X \in LC(G)$, so $\forall t \in \mathbb{R} : exp(tX) \in C(G)$. Let $Y \in \mathfrak{g}$, then:

$$Y = Ad_{exp(tX)}(Y) \underset{com.diag.}{=} exp(ad_{tX})(Y) = e^{ad_{tX}}(Y) \Rightarrow e^{ad_{tX}} = Id$$

Chossing $t > 0$ small enough, this implies $ad_{tX} = 0$, i.e. $t \cdot ad_X = 0$, i.e. $ad_X = 0$. This means $X \in C(\mathfrak{g})$ □

Let G be a Lie group, $H \subset G$ closed Lie group. Let ${}^{G}/_{H} := \{aH | a \in G\}$ be the set of left-cosets of H.

Theorem 2.14. *${}^{G}/_{H}$ has the structure of smooth manifold of dimension $dim(G) - dim(H)$ and such that:*

$$\pi : G \to {}^{G}/_{H} \quad a \mapsto aH$$

is a smooth map that admits local smooth section.

Remark 2.15. ${}^{G}/_{H}$ is called a **homogeneous space** for C.

Proof. Step 1: Topological observations
${}^{G}/_{H}$ is given the quotient topology, defined by $U \subset {}^{G}/_{H}$ is open $\Leftrightarrow \pi^{-1}(U) \subset G$ is open.

- $\pi : G \to {}^{G}/_{H}$ is an open map: let $U \subset G$ open, then $\pi^{-1}(\pi(U)) = U \cdot H = \bigcup_{k \in H} U \cdot k$ is open, hence so is $\pi(U)$.
- ${}^{G}/_{H}$ has a countable basis for its topology. If $\mathfrak{B}$ is a basis for the top. on G, then $\{\pi(B) | B \in \mathfrak{B}\}$ is a basis for top. on ${}^{G}/_{H}$.
- ${}^{G}/_{H}$ is Hausdorff. Suppose $aH \neq bH$ i.e. $a^{-1}bH \neq H$ i.e. $a^{-1}b \notin H$.

Consider

$$f : G \times G \to G \quad (g_1, g_2) \mapsto g_1^{-1} g_2$$

By continouty of f ad the assumption that H be closed. $f^{-1}(H) \subset G \times G$ is closed we have $(b, a) \in (G \times G) - f^{-1}(H)$, i.e. there are open sets $U, V \subset G$ s.t. $b \in U, a \in V$ and $(U \times V) \cap f^{-1}(H) = \emptyset$. Then $\pi(U)$ and $\pi(V)$ are disjoint open nbhd. of bH and aH. respectivly. Step 2: Constructions of the coodinate charts
Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k} \subset \mathfrak{g}$ the Lie subalgebra corresponding to H. Choose a complement if $\mathfrak{k}$ in $\mathfrak{g}$ (as vector spaces) : $\mathfrak{g} \oplus \mathfrak{h} = \mathfrak{g}$.Consider the smooth map:

$$\varphi : \mathfrak{k} \oplus \mathfrak{h} \to G \quad (X, Y) \mapsto exp(X) \cdot exp(Y)$$

CLaim:" $D_{(0,0)}\varphi$ is an isomorphism"

Proof. Let $X \in \mathfrak{k}$ and $\gamma(t) = \varphi(tX, 0) = exp(tX)$. Then $(D_{(0,0)}\varphi)(X, 0) = \frac{d}{dt}|_{t=0} exp(tX) = X$i.e. $Im(D_{(0,0)}\varphi) \supset \mathfrak{k}$. Similarly, $Im(D_{(0,0)}\varphi) \supset \mathfrak{g}$. Since $dim(G) = dim(\mathfrak{g}) = dim(\mathfrak{k} \oplus \mathfrak{h}), D_{(0,0)}\varphi$ is an isomorphism. □

CLaim:"$V_1 \subset \mathfrak{k}$ and $V_2 \subset \mathfrak{h}$ be open nbhd.around 0, such that $\varphi : V_1 \times V_2 \to U$ is a diffeomorphism onto an open nbhd. U of $e \in G$. Then there is an open set $U_1 \subset V_1$s.t.:$((expU_1)^{-1} exp(V_2)) \cap H \subset exp(V_2)$"

Proof. Since $exp(V_2)$ is an open nbhd. of $e \in G$ and H has the subspace topology of G, there exists an open nbhd. W of e in G s.t. $exp(V_2) = W \cap H$. Now choose U_1 s.t. $exp(U_1)^{-1}exp(V_2) \subset W$. Lets assume w.l.o.g. that $V_1 = U$. □

Claim:"Define $\phi_e : V_1 \to \pi(U)$ by $\phi_e(X) := exp(X) \cdot H$. Then ϕ_e is bijective."

Proof. ϕ_e is surjective: By def $\varphi(V_1 \times V_2) = exp(V_1)\underbrace{exp(V_2)}_{\subset H}$, so $\pi(U) = \pi(exp(V_1)) = \phi_e(V_1)$.
ϕ_e is injective: Suppose $\phi_e(X) = \varphi_e(Y)$ i.e. $exp(X)H = exp(Y)H$, for some $X, Y \in V_2$. Then

$$exp(X)^{-1}exp(Y) \in (exp(V_1)^{-1}exp(V_2)) \cap H \subset exp(V_2)$$

So there is some $Z \in V_2$s.t. $\varphi(Y, 0) = exp(Y) = exp(X)exp(Z) = \varphi(X, Z) i.e. Y = X$. □

For each $a \in G$ define a chart around $aH \in {}^{G}/_{H}$ by $U_2 = a \cdot \pi(U), \phi_e : V_1 \to a\pi(U), \phi_a := l_a \circ \phi_e$.
Claim:"The transition map $\phi_b^{-1}\phi_a : \phi_e^{-1}(U_a \cap U_b) \to \phi_b^{-1}(U_a \cap U_b)$ are smooth"

Proof. We'll write $\phi_b^{-1}\phi_a$ as a composition of smooth maps. Let $p \in U_a \cap U_b$ then:

$$p = aexp(X)H = bexp(Y)H \quad \text{for some } X, Y \in V_1$$

Note $(\phi_b^{-1}\phi_a)(X) = Y$. Hence $\exists H \in H : exp(X)h = \underbrace{(a^{-1}b)}_{\in l_{a^{-1}b}(U) \subset G open} \cdot exp(Y)$.
By continouty of multiplications, there is an open nbhd. A and X in $X \in \phi_a^{-1}(U_a \cap U_b)$ s.t. $exp(A)h \subset l_{a^{-1}b}(U)$. □

Deinfe smooth maps:

$$\nu : A \to l_{a^{-1}b}(U) \quad X' \mapsto exp(X')h$$

$$p : l_{a^{-1}b}(U) \to V_1, p := \pi_1 \circ \varphi^{-1} \circ l_{b^{-1}a}$$

Claim:"$\phi_b^{-1}\phi_a|_A = p \circ \nu$"

Proof. Let $X' \in A$. Then

$$\nu(X') = \underbrace{exp(X')h}_{\in l_{a^{-1}b}(U)}(a^{-1}b)exp(Y')\underbrace{exp(Z')}_{\in H}$$

With uniquely determined $(Y'Z') \in V_1 \times V_2$. $\Rightarrow aexp(X')H = bexp(Y')H$ i.e. $(\phi_b^{-1}\phi_e)(X') = Y'$. Easy to check also $p(\nu(X')) = Y'$ also. □

Claim:"$\pi : G \to {}^{G}/_{H}$ is smooth"

Proof. It suffices to check that $\pi|_U : U \to \pi(U)$ is smooth. Can easily check that the following diagram commutes:

$$\begin{array}{ccc} V_1 \times V_2 & \xrightarrow{\varphi} & U \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi|_U} \\ V_1 & \xrightarrow{\phi_e} & \pi(U) \end{array}$$

i.e. $\phi_e^{-1} \circ \pi \circ \varphi = \pi_1$. Since π_1 is smooth, so is $\pi|_U$. The diagram also shows that $U \to \pi(U)$ has a smooth section becouse π_1 does. □

□

Prime example of a principle H-bundle:

$$\begin{array}{ccc} V_1 \times H & \xrightarrow{*} & G \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \cdot} \\ V_1 & \xrightarrow{\phi_e} & {}^{G}/_{H} \end{array}$$

$* = (X, h) \to exp(X)h$
Vorlesung 6:

3 Principal Bundles

G Lie group, M a smooth manifold.

Definition 3.1. A smooth **principal G-bundel** over M is a smooth manifold P together with the following:

(1) a smooth map $\pi : P \to M$

(2) a smooth right G-action on P:

$$P \times G \to P \quad (p, g) \mapsto \phi_g(p) = p \cdot g$$

satisfying:

(a) the G-action preserves the fibres of πi.e.: $\pi(p \cdot g) = \pi(p) \forall p \in P g \in G$

(b) there is a covering of M by open sets $\{U_\alpha\}_{\alpha \in I}$ together with diffeomorphism:

$$\Phi_\alpha : \pi^{-1}(U_\alpha) \underset{\cong}{\to} U_\alpha \times G \quad p \mapsto (\pi(p), \phi_\alpha(p))$$

such that $\phi_\alpha(p \cdot g) = \phi_\alpha(p) \cdot g \quad \forall p \in \pi^{-1}(U_\alpha) g \in G$.

[i.e. ϕ_α is G-equivalent, wrt. the ... right G-action on the secound factor of $(U_\alpha \times G), f : X \to Y$ **map of G-spaces is G-equiv.**, if $f(x \cdot g) = f(x) \cdot g \forall x \in X g \in G$]

Terminologie:

- P total space
- M base space
- G structure group
- π projection map
- Φ_α local trivialisations
- $\{U_\alpha\}_{\alpha \in I}$ trivialising open cover of M

Remark 3.2. Suppose $P \underset{\pi}{\to} M, (P, \pi, M), G \hookrightarrow P$ is a principal G-bundle.

(i) π admits local smooth sections: by local triviality:

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\Phi_\alpha} & U_\alpha \times G \\ & {\scriptstyle \pi|_{\pi^{-1}(U_\alpha)}} \searrow \quad \swarrow {\scriptstyle \pi_1} & \\ & U_\alpha & \end{array}$$

but π_1 has smooth sections, namely:

$$s : U_\alpha \to U_\alpha \times G \quad x \mapsto (x, g)$$

for some fixed $g \in G$.[more generally, can take any smooth function $t : U_\alpha \to G$ then $x \mapsto (x, t(x))$ is a section] But the $\phi_\alpha^{-1} \circ s$ is a section of $\pi|_{\pi^{-1}(U_\alpha)} \to U_\alpha$

(ii) π is always surjective submersion: Surjektivity follows from (b), since the U_α cover H. Submersion: becouse locally it is diffeomorphic to $\pi_1 : U_\alpha \times G \to U_\alpha$, which is a submersion.

(iii) $\pi^{-1}(x)$ is diffeomorphic to G for all $x \in M$. Thus follows from local triviality, by restricting to fibres.

(iv) On each fibre $\pi^{-1}(x)$ the G-action is **simply transitive**(free and transitive) i.e.: transitive with trivial .... This follows from $\pi^{-1}(x) \cong G$ compatibly with the G-action and the fact that the right-action of G on itself is simply transitive.

(v) M is homeomorphic with P/G. Because the aciton of G on P is fiberwise, the map $\pi : P \to M$ factors through $P/G \to M$. This is an open, continous bijection i.e. a homeomorphism.

Example 3.3. *(1) $P = M \times G$ with $\pi = \pi_1$ the projection onto M and the obvious rigth action of G on the secound factor, is a principle G-bundle for any smooth mfd. M and Lie group G. H is called the* ***product bundle****.*

(2) G a Lie group, $H \subset G$ a closed subgroup. Then the canonical projection $\pi : G \to G/H$ together with the obvies H-action on the right of G, is a principle H-bundle by yesterday's lecture. We showed that G/H can be covered by charts $U_g, g \in G$ such that:

$$\begin{array}{ccc} V_1 \times H & \xrightarrow{*} & \pi^{-1}(U_g) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi} \\ V_1 & \xrightarrow{\varphi_g} & G/H \end{array}$$

(*): $\Phi^{-1}, \cong$. Commute, where $\Phi^{-1}(X,h) = exp(X) \cdot k$.($V_1 \subset \mathfrak{k}$ open nbhd. of 0, $\mathfrak{k} \oplus \mathfrak{h} = \mathfrak{g}$). Since

$$\Phi_g^{-1}((X,h)\cdot h') = \Phi_g^{-1}(X,hh') - exp(X)\cdot h \cdot h' = (\Phi_g^{-1}(X,h)) \cdot h'$$

we see that Φ_g^{-1} is H-equivalent hence, Φ_g is also H-equivalent.

(3) M is a smooth n-manifold, P the set of bases for the tangent space $T_xM, x \in M$.

$$\pi : P \to M \quad (v_1, ..., v_n) \in T_xM \mapsto x$$

P has a smooth manifold structure so that $P \underset{\pi}{\to} M$ is a smooth principle $GL_n(\mathbb{R})$-bundle. This is called the tangent frame bundle of M.

Definition 3.4. Let $\pi : P \to M$ and $\pi' : P' \to M$ be two principle G-bundles over the same .... manifold M. An **isomorphism between P and P'** is a diffeomorphism $f : P \underset{\cong}{\to} P'$ satisfyng:

- $\pi' \circ f = \pi$ i.e. the following diagram commutes:

$$\begin{array}{ccc} P & \xrightarrow{f} & P' \\ & {\scriptstyle \pi}\searrow \quad \swarrow{\scriptstyle \pi'} & \\ & M & \end{array}$$

- $f \circ r_g = r_g \circ f \quad \forall g \in G$ i.e. f is G-equivalent.
- clearly, f is smooth
- for $h \in G$ we have : $f(x,g) \cdot h) = f(x, gh) = s(x) \cdot (gh) = f(x,g) \cdot h)$ i.e. f is equivalent.
- $\pi(f(x,g)) = \pi(s(x) \cdot g) = \pi(s(x)) = x = \pi_1(x,g))$ so f is a map compatible with the projections to M.
- f maps $\{x\} \times G$ bijectivly onto $\pi^{-1}(x)$, since f is G-equivalent and G acts simply transitivly on the fibres (e.g. $f(x,g) = f(x,g') \Leftrightarrow s(x) \cdot g = s(x) \cdot g' \Leftrightarrow s(x) = s(x)(g' \cdot g^{-1}) \Leftrightarrow g' \cdot g^{-1} \in G_{s(x)} = \{e\} \Leftrightarrow g = g'$)

Lemma 3.5. *A principle G-bundle P admits a global smooth section $s : M \to P \Leftrightarrow$ it is isomorphic with the product bundle $\pi_1 : M \times G \to M$.*

Proof. "$\Leftarrow$" The product bundle has a section $s : M \to M \times G \quad x \mapsto (x, e)$ If $f : M \times G \to P$ is an isomorphism, $f^{-1} \circ s : M \to P$ is a section for π.
"$\Rightarrow$" Conversely, suppose P admits a globa section $s : M \to P$. Define

$$f : M \times G \to P \quad (x,g) \mapsto s(x) \cdot g$$

It remains to show that f^{-1} is smooth. Let $U \subset M$ be open such that $\pi|_{\pi^{-1}(U)} : \pi^{-1}(U) \to U$ is trivial.

$$\begin{aligned} U \times G \underset{f|_{U\times G}}{\to} \pi^{-1}(U) &\underset{\Phi,\cong}{\to} U \times G \\ (x,g) \mapsto s(x) \cdot g &\mapsto (x, \varphi(s(x))) = (x, t(x) \cdot g \end{aligned}$$

Since φ is G-equivalent, $\varphi(s(x) \cdot g) = t(x) \cdot g$ for some smooth $t : U \to G$. Thus, the inverse is given by $(x,g) \mapsto (x, t(x)^{-1} \cdot g)$. This is a smooth map. □

Terminology: P is called (globally) trivial if it is isomorphic to the product bundle.
Vorlesung 7:
G a Lie group $g \in G$:

$$a_g : G \to G \quad p \mapsto gpg^{-1}$$

$Ad : G \to GL(T_eG)$

Lemma 3.6. *$Ad(g)$ is given by the compostition of $D_e r_{g^{-1}}$ with the identification $T_{g^{-1}}G \to T_eG$ given by left multiplication (with g).*

Proof. Consider a $v \in T_{g^{-1}}G$. There is a unique left-invariant vector field $X \in \mathfrak{g}$ s.t. $X(g^{-1}) = v$. Then $f(v) = X(e) \Leftrightarrow D_{g^{-1}}l_g(v) = X(e)$.

$$\begin{aligned} Ad(g)(Y(e)) &= D_g r_{g^{-1}} \circ D_e l_g(Y(e)) = D_g r_{g^{-1}}(Y(g)) \\ &= D_{g^{-1}} l_g \circ \underbrace{D_e r_{g^{-1}}(Y(e))}_{v} = X(e) \end{aligned}$$

□

$P \to M$ a principle G-bundle, $P \times G \to P$ the right G-action. Each $X \in \mathfrak{g}$ determines a 1-par. subgroup $exp(tX)$. By restriction of the right G-action, $exp(tX)$ defines a flow on P:

$$\mathbb{R} \times P \to P \quad (t, p) \mapsto p \cdot exp(tX)$$

Definition 3.7. X^* is the vector field on P obtained by differentiating this flow. X^* is called the **fundamental vector field** defined by $X \in \mathfrak{g}$.

Properties:

(a) $X^* \in ker(D\pi)$

(b) $X^*(p) \neq 0 \forall p \in P if X \neq 0$

(c) $\mathfrak{g} \to \mathfrak{X}(p)$ / $X \to X^*$ is an injective homomorphism.

(d) At every point $p \in P$, the values of $X^*(p)$ as X varies in $\mathfrak{g}$ from $ker(D_p\pi)$

Lemma 3.8. *For $g \in G$ and $X \in \mathfrak{g}$ we have:*

$$D_p r_g(X^*(p)) = (Ad(g^{-1})X)^*(p \cdot g)$$

Proof. Following diagram commutes:

$$\begin{array}{ccc} G & \xrightarrow{a_{g^{-1}}} & G \\ \uparrow{\scriptstyle exp} & & \uparrow{\scriptstyle exp} \\ \mathfrak{g} & \xrightarrow{Ad(g^{-1}): X \mapsto Y} & \mathfrak{g} \end{array}$$

$exp(tY) = g^{-1}exp(tX)g$ Define $s, s' : \mathbb{R} \to P$ by $s(t) = p \cdot exp(tX)$ and $s'(t) = p \cdot g \cdot exp(tY) = pexp(tX)g$. $D_p r_g(X^*(p)) = D_p r_g \circ D_0 s(\frac{\partial}{\partial t}) = D_0 s'(\frac{\partial}{\partial t}) = Y^*(p \cdot g) = (Ad(g^{-1})X)^*(p \cdot g)$ □

$\pi : P \to M$ a principle G-bundle. There is a covering of M by open sets $U_\alpha, \alpha \in I$s.d.

$$\pi^{-1}(U_\alpha \to U_\alpha) \times G \quad p \mapsto (\pi(p), \varphi_\alpha(p)) \quad \varphi_\alpha(p \cdot g) = \varphi_\alpha(p) \cdot g$$

$$(x,g) \longrightarrow (x, \bar{\psi}_{\alpha,\beta}(x,g))$$

$$\begin{array}{ccc} (U_\alpha \cap U_\beta) \times G & \xrightarrow{Id\times ?} & (U_\alpha \cap U_\beta) \times G \\ & \searrow^{(\pi\times\varphi_\beta)^{-1}} \quad \nearrow^{(\pi\times\varphi_\alpha)} & \\ & M & \end{array}$$

Define $\psi_{\alpha,\beta}(x) = \bar{\psi}_{,\beta}(x,e)$ Then $ps\bar{i}_{\alpha,\beta}(x,g) = \psi(x)\cdot g \quad \forall g \in G$.

$$\psi_{\alpha,\beta} : (U_\alpha \cap U_\beta) \to G \quad \text{are smooth maps}$$

Properties:

(1) $\psi_{\alpha,\alpha} : U_\alpha \to G$ is the constant map to $e \in G$

(2) $\psi_{\beta,\alpha}(x) = \psi^{-1}_{\alpha,\beta}(x) \quad \forall x \in U_\alpha \cap U_\beta$

(3) if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, then

$$\psi_{\alpha,\beta}(x)\cdot\psi_{\alpha,\gamma}(x) = \psi_{\alpha,\gamma}(x) \quad \forall x \in U_\alpha \cap U_\beta \cap U_\gamma$$

These properties are the **cocycle conditions** for the maps $\psi_{*,*}$

One can reconstruct P (up to isomorphism) just from the cocycle $\psi_{*,*}$

$$(\coprod_{\alpha\in I}(U_\alpha \times G))/\simeq\; = P$$

$(x,g) \in U_\alpha \times G$ is equivalent to $(y,h) \in U_\beta \times G$ if $x = y$ in M and $h = \psi_{\alpha,\beta}(x)\cdot g$. This is an equivalence relation because $\psi_{*,*}$ satisfies the cocyle conditions. P is smooth mfd. with local parametrization given by the projections of $U_\alpha \times G$ to P. Define :

$$P \times G \to P \quad ([(x,g)], g') \mapsto [(x, gg')]$$

This is a well-def., smooth right G-action on P. This action is free and the space of orbits is M.
Let:

$$\begin{array}{ccc} P & \xrightarrow{f_1} & P' \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ M & \xrightarrow{\bar{f}_1} & M' \end{array}$$

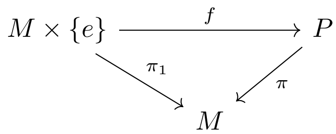
be a principle (G-)G'-bundles.

Definition 3.9. A **homomorphism of principle bundles** is a pair of smooth maps $f_1 : P \to P', f_2 : G \to G'$ s.t.: f_2 is a Lie group homomorphism and $f_1(p \cdot g) = f_1(p) \cdot f_2(g) \quad \forall p \in Pg \in G$

Notation: We usually write f for both f_1 and f_2. We often consider the case $\bar{f} = Id_M$. Let $\pi : P \to M$ be a principle G-bundle and $H \subset G$ a Lie subgroup.

Definition 3.10. A reduction of the structure group of P from G to H is a princ. H-bundle $P' \to M$ together with an injective homomorphism $f : P' \to P$ (with $\bar{f} = Id_M$). s.t. $f_2 : H \to G$ is the inclusion.

Example 3.11. *$H = \{e\}$ then $P' = M \times \{e\}$ and an injective homomorphism to $P \Rightarrow P$ has a section defined by $s(x) = f(x, e)$.*

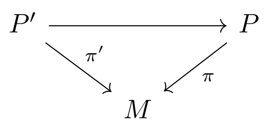


A princ. G-bundle $\pi : P \to M$ has a reduction of the structure group to $\{e\} \Leftrightarrow$ it is globally trivial.

Vorlesung 8:

Proposition 3.12. *A princ. G-bundle $\pi : P \to M$ has a reduction to $H \subset G$ if and only if there is a system of lo. trivializations for P s.t. the corresponding cocyle $\psi_{\alpha,\beta} : U_\alpha \cap U_\beta \to G$ takes values in H.*

Proof. "$\Leftarrow$":Suppose we have a coclye $\psi_{*,*}$ for P s.t. the values are in $H \subset G$. then there is a princ. H-bundle P' defined by $\psi_{*,*}$. Moreover there is an inclusion $P' \hookrightarrow P$, which is a homomorphism of princ. bundles. $\Rightarrow P'$ is a reduction of the structure group of P to H.
"$\Rightarrow$": Suppose we have a reduction:



Take any system of loc. trivialization for f', then the corresponding $\psi_{*,*}$ takes values in H. Composing: $U_\alpha \cap U_\beta \underset{\psi_{\alpha,\beta}}{\to} H \hookrightarrow G$ we can think of $\psi_{*,*}$ as a cocyle with values in G. This is a cocyle for P with values in H □

4 Associated bundles

Let $\pi : P \to M$ be a princ. G-bundle and $G \times F \to F$ a smooth G-action on the left on some smooth mfd. F. Associated to this data is $E := P \underset{G}{\times} F := {}^{(P \times F)}/_{G} = {}^{(P \times F)}/_{\simeq}$ where

$$(p, f) \simeq (p \cdot g, g^{-1} f) \quad \forall g \in G$$

Claim:"$(P \times F) \times G \to P \times F, ((p, f), g) \mapsto (p \cdot g, g^{-1} f)$ defines a smooth right G- action on $P \times F$"
We will see that E is a smooth mfd. with a smooth submersion $\pi_E : E \to M$ s.t. $\pi_E^{-1}(x)$ is diffeomorphic to F for every $x \in M$. Let $U \subset M$ be an open set over which P is trivial. Pick a loc. trivialization:

$$\pi^{-1}(U) \to U \times G \quad p \mapsto (\pi(p), \varphi(p))$$

The construction of E with M replaced by U and P replaced by $\pi^{-1}(U)$ gives:

$$\pi_E^{-1}(U) = {}^{(\pi^{-1}(U) \times F}/_{G} = {}^{(U \times G \times F}/_{G}$$

with G acting as follows: $(x, g', f) \cdot g = (x, g'g, g^{-1} \cdot f)$. Define:

$$\sigma : {}^{(U \times G \times F)}/_{G} \to U \times F \quad [(x, g', f)] \mapsto (x, g' \cdot f)$$

Suppose $(x, g', f) \simeq (x, g'', f')$. Then there exists a $g \in G$ s.t.

$$g'' = g' \cdot g \quad f' = g^{-1} \cdot f$$
$$\Rightarrow g'' \cdot f' = (g'g)(g^{-1} f) = (g'gg^{-1}) \cdot f = g' \cdot f$$

Define:

$$\tau : U \times F \to {}^{(U \times G \times \times F)}/_{G} \quad (x, f) \mapsto [(x, e, f)]$$
$$\sigma \circ \tau(x, f) = \sigma([x, e, f)]) = (x, f)$$
$$\tau \circ \sigma([x, g', f]) = \tau(x, g'f) = [(x, e, g'f)] = [(x, g', f)]$$

We give E the mfd. structure which makes each $\pi^{-1}(U)$ an open set diffeomorphic to $U \times F$ then define $\pi_E : E \to M, [(p, f)] \mapsto \pi(p)$. In the charts for E constructed above, π_E is:

$$\pi_1 : U \times F \to U \quad ((p, f), g) \mapsto (p \cdot g, g^{-1} f)$$

Therefore, π_E is smooth, a submersion, and $\pi_E^{-1}(x) \cong F \forall x$.

Example 4.1. *Fix $\pi : P \to M$:*

(0) Take $G \times F \to F$ the trivial G-action with $g \cdot f = f \quad \forall g \in G, f \in F$. Then $E = P \underset{G}{\times} F = M \times F$ with $\pi_E = \pi_1$.

(1) *Let $\rho : G \to GL(V)$ be a linear rep. of G. This defines a left action of G on V via $G \times V \to V, (g, v) \mapsto \rho(g) \cdot v$. Then $E = P \underset{G}{\times} G$ is a vector bundle over M with fiber V.*<u>*Notation:*</u>*E is denoted $P \underset{\rho}{\times} V$*
Converse construction: start with a rank k vector bundle $E \to M$. The set of local frames for the fibers of E form a $Gl_k(\mathbb{R})$-principal bundle $P \to M$. Now take $\rho = Id$. Then the vector bundle associated to P via ρ is isomorphic to E.
Special case: $\rho = Ad : G \to GL(\mathfrak{g})$. Then $E = P \underset{Ad}{\times} \mathfrak{g}$. Is the adjoint bundle P denoted $Ad(P) = P \underset{Ad}{\times} \mathfrak{g}$

(2) *$H \subset G$ a closed Lie subgroup. Then G acts on the left on the space ${}^{G}/_{H}$:*

$$G \times {}^{G}/_{H} \to {}^{G}/_{H} \quad (g, g'H) \to gg'H$$

Taking $F = {}^{G}/_{H}$ with this G-action we have an associated bundle $E = {}^{(P \times {}^{G}/_{H})}/_{G}$ over M with fiber ${}^{G}/_{H}$.

Lemma 4.2. *This $E = {}^{(P \times {}^{G}/_{H})}/_{G}$ is diffeomorphic to ${}^{P}/_{H}$, where $H \subset G$ acts on the right via the p.b.action.*

Proof. Define:

$$\nu : E \to {}^{P}/_{H} \quad [(p, gH)] \mapsto [p \cdot g]$$

If $(p, gH) \simeq (q, g'H)$ then there is a $\alpha \in G$ such that $p \cdot \alpha = q$ and $\alpha^{-1} gH = g'H \Leftrightarrow (g')^{-1}\alpha^{-1} g \in H \Leftrightarrow g^{-1\prime} \in H$. $[p \cdot g] = [(p \cdot g) \cdot (g^{-1}\alpha g')] = [p \cdot (\alpha g')] = [q \cdot g']$ so ν is well-def. Define:

$$\tau : {}^{P}/_{H} \to E \quad [p] \mapsto [p, e \cdot H)]$$

This is also well-defined. ν and τ are inverses of each other. They are smooth because they are smooth in loc. trivializations. ☐

<u>*Vorlesung 9:*</u>

$$G \times {}^{G}/_{H} \to {}^{G}/_{H}$$

$$E = P \underset{G}{\times} ({}^{G}/_{H}) = {}^{P}/_{H}$$

Proposition 4.3. *P admits a reduction to the subgroup $H \subset G \Leftrightarrow E$ has a section*

Proof. "$\Rightarrow$":Suppoe P' is a princ. H-bundle and $f : P' \to P$ is a reduction.

$$P' \underset{f}{\to} P \underset{g}{\to} {}^{P}/_{H} = E$$

g the projection.
Claim:"$g \circ f$ is constant on every fibre of P'"

Proof. Suppose $p, g \in P'$ are in the same fibre. Then $\exists h \in H$ s.t. $q = p \cdot h \Rightarrow f(q) = f(p) \cdot h \Rightarrow gf(q) = gf(p)$. $\square$

Define:

$$s : M \to E \quad s(x) = gf(p) \quad \text{for any } p \in \pi^{-1}(x)$$

This is a smooth section of E. "$\Leftarrow$": Suppose E has a section $s : M \to E$. Consider the projection $g : P \to {}^{P}/_{H} = E$ and define $P' = g^{-1}s(M)$. Over every point $x \in M$ we have:

$$G \underset{g}{\to} {}^{G}/_{H} \quad g^{-1}s(x) \mapsto s(x)$$

Claim:"P' is a princ.H-bundle"

Proof. $g^{-1}s(M) \subset P$ .G acts on P by the princ. bundle action and we rstrict this action to $H \subset G$. If $p \in g^{-1}(s(M))$ and $h \in H$, then $p \cdot g \in g^{-1}(s(M))$ the right action by H maps $g^{-1}(s(M))$to itself and acts simply transitively on its fibres over M. So $P' = g^{-1}(s(M))$ is a princ. H-bundle giving a reduction of P to H. $\square$

$\square$

5 Connections

Let $\pi : P \to M$ be a princ. G-bundle. $V := kerD\pi \subset TP$ is a smooth subbundle called **vertical bundle**.

Definition 5.1. A **connection on P** is a choice of complementary subbundle $H \subset TP$ that is invariant under the right G-action, i.e. $D_p r_g(H_p) = H_{p \cdot g}$

Terminology: H is called a **horizontal bundle**(Connexion = connection)
Reminder: $\alpha \in \Omega^1(M)$ s.t. $\alpha(p) \neq 0 \forall p \in M \Rightarrow \alpha(p) \neq 0 \in T^*M \Rightarrow ker\alpha$ is a smooth subbundle of TM of codim 1. If $\alpha \in \Omega^1(M, V)$ a smooth 1-form with values in V, V a $\mathbb{R}$-vector space, then $\alpha(p) : T_pM \to V$, $\alpha(p) \in T_p^*M \bigotimes V$. If $\alpha(p)$ is surjective onto V, then $ker\alpha$ is a subspace of codimension equal to the dimension of V. Whenever this is true for all $p \in M$, $ker\alpha$ is a smooth subbundle of TM of codimension equal to dim V.
Given a connection H on $\pi : P \to M$ we define $w \in \Omega^1(P; \mathfrak{g})$ by:

$$w(X) = 0 \text{ for } X \in H \quad w(X_p) = Y \text{ if } X_p = Y_p^* \text{ for some } Y \in \mathfrak{g}$$

Terminology:w has $kerw = H$ and is called **the connection 1-form** for H.
Claim: " w is well-def."

Proof.
$$T_pP = V_p \oplus H_p \quad V_p = kerD\pi = span\{Y^*|Y \in \mathfrak{g}\}$$
□

Lemma 5.2. $r_g^* w = Ad(g^{-1})w$

Proof.
$$r_g^* w_p(X_p) = w_{p\cdot g}(D_p r_g(X_p))$$

Both sides of the Claim are linear, so it is enough to check seperately the values on horizontal and vertical tangent vectors. Takes $X_p \in H_p$. Then $w_p(X_p) = 0 \Rightarrow Ad(g^{-1}(w_p(X_p)) = 0$

$$(r_g^* w)(X_p) = w_{p\cdot g}(\underbrace{D_p r_g(X_p)}_{\in H_{p\cdot g}}) = 0$$

Take $X_p \in V_p : \exists ! Y \in \mathfrak{g}$ s.t.$Y_p^* = X_p$.

$$\begin{aligned} (r_g^* w)(X_p) &= w_{p\cdot g}(D_p r_g(X_p)) = w_{p\cdot g}(D_p r_g(Y_p^*)) \\ &= W_{p\cdot g}(Z_{p\cdot g}^*) = Z = Ad(g^{-1})Y \\ Ad(g^{-1})w(X_p) &= Ad(g^{-1})w_p(Y_p^*) = Ad(g^{-1})Y \end{aligned}$$

where $Z = Ad(g^{-1})Y$ □

Conversly, suppose we have an arbitrary $w \in \Omega^1(P; \mathfrak{g})$ satisfying $r_g^* w = Ad(g^{-1})w$ and $w(Y^*) = Y \quad \forall Y \in \mathfrak{g}$

Lemma 5.3. *$kerw$ is a connection on P.*

Proof. dim T_pP = dim M + dim G, dim $\mathfrak{g}$ = dim G. At every point $p \in P$, $w_p : T_pP \to \mathfrak{g}$ is surjective $\Rightarrow kerw_p$ has dim = dim M $\forall p \in P \Rightarrow H := kerw \subset TP$ is a smooth subbundle.

$$\begin{array}{ccc} w_p : T_pP & \longrightarrow & \mathfrak{g} \\ & \overset{\cong}{\nearrow} & \\ V_p \subset T_pP & & \end{array}$$

$$Y_p^* \mapsto Y$$

$\Rightarrow H_p = kerw_p$ is a complement to V_p, so H is complementary to V. Suppose $X_p \in H_p$. Then $w_{p\cdot g}(D_p r_g(X_p)) = r_g^* w(X_p) = Ad(g^{-1})\underbrace{w_p(X_p)}_{=0} = 0$ implies

$D_p r_g(X_p) \in H_{p\cdot g}$ □

Definition 5.4. $w \in \Omega^1(P, \mathfrak{g})$ is called a **connection 1-form** if $w(Y_p^*) = Y$ $\forall Y \in \mathfrak{g}$ and $r_g^* w = Ad(g^{-1})w \quad \forall g \in G$. There is a 1-1 correspondence between connection H and connection 1-forms w :

$$\{H\} \longrightarrow \{w\} \longrightarrow \{H\}$$

$$w \longrightarrow ker(w)$$

Proposition 5.5. *Every princ. G-bundle $\pi : P \to M$ has a connection 1 form*

Proof. Let U_α be open sets in M s.t. $\pi^{-1}(U_\alpha)$ is trivial: $\pi^{-1}(U_\alpha) \to U_\alpha \times G$ has a connection with $H_{(x,g)} = T_x U_\alpha \times \{0\}$. Let w_α be the corresponding connection 1-form in $\Omega^1(\pi^{-1}(U_\alpha), \mathfrak{g})$. We may cover M by such U_α and take a partiion of unity ρ_α:

$$supp \rho_\alpha \subset U_\alpha \quad \rho_\alpha \in C^\infty(M) \quad \rho_\alpha \geq 0 \quad \sum_\alpha \rho_\alpha \equiv 1$$

Define $w := \sum_\alpha \pi^*(\rho_\alpha) \cdot w_\alpha = \sum_\alpha (\rho_\alpha \circ \pi) w_\alpha$
Claim: "w is a connection 1-form on P"

Proof. to prove:

(1) $w : T_p M \to \mathfrak{g}$ is surjective $\forall p \in P$

(2) $ker(w)$ is complementary to V

(3) $ker(w)$ is preserved by r_g

where $w(Y_p^*) = Y \quad \forall Y \in \mathfrak{g}, r_g^* w = Ad(g^{-1})Y$
Vorlesung 10:

$$\begin{aligned} r_g^* w &= \sum_\alpha (\rho_\alpha \circ \pi) r_g^* w_\alpha = \sum_\alpha (\rho_\alpha \circ \pi) Ad(g^{-1}) w_\alpha \\ &= Ad(g^{-1}) \sum_\alpha (\rho_\alpha \circ \pi) w_\alpha = Ad(g^{-1}) w \\ w(Y^*) &= \sum_\alpha \pi^*(\rho_\alpha) w_\alpha (Y^*) \\ &= \sum_\alpha \pi^*(\rho_\alpha) Y = Y \quad \forall Y \in \mathfrak{g} \end{aligned}$$

So w is a connection 1-form on P ∎

∎

Theorem 5.6. *The set of connections on P is actually an affine space for the vector space* $\Omega^1(M, Ad(P)) = \Gamma(T^*M \otimes Ad(P))$ *where* $Ad(P) = P \underset{Ad}{\times} \mathfrak{g}$.

Proof. Fix a connection w_0 on P. As w_1 ranges over all connection on P, we want to prove that $w_1 - w_0 = \tilde{w}$ ranges over $\Omega^1(M, Ad(P))$. If $s \in \Omega^1(M, Ad(P))$ and $Y_* \in TM$ $x \in M$, then $s(y) = [(p, Z)]$ for some $Z \in \mathfrak{g}$ $\pi(p) = x$.

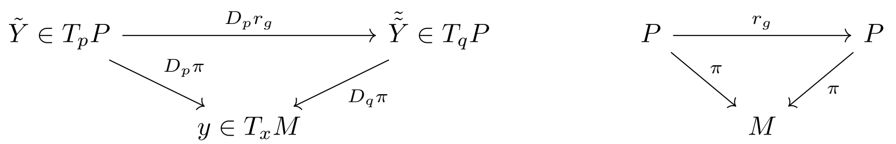
$$Ad(P) = {}^{(P \times \mathfrak{g})}/_{\simeq} \text{ with } (p, Z) \simeq (p \cdot g, Ad(g^{-1}Z)) \quad \forall g \in G$$

$$\tilde{w}(X^*) = w_1(X^* - w_0(X^*) = X - X = 0 \Rightarrow V \subset ker(\tilde{w})$$

Take $p \in \pi^{-1}(x)$, consider $D_p\pi : T_pP \to T_xM$. This is surjective, so $\exists \tilde{Y} \in T_pP$ s.t.$D_p\pi(\tilde{Y}) = Y$. Consider $\tilde{w}(\tilde{Y}) \in \mathfrak{g}$. This is independent of the choice of lift $\tilde{Y}$ for Y, because any two lifts differ by a vector in $V \subset ker(\tilde{w})$. Define a section $s \in \Gamma(T^*M \otimes Ad(P))$ by:

$$s_x(Y_x) = [(p, \tilde{w}(\tilde{Y}))]$$

Take another $q \in \pi^{-1}(x)$. Then $\exists g \in G$s.t. $q = p \cdot g$ We may take $\tilde{\tilde{Y}} = (D_pr_g)(\tilde{Y})$ as a lift of Y to T_qP.



$$\begin{aligned}(q, \tilde{w}(\tilde{\tilde{Y}})) &= (p \cdot g, \tilde{w}(Dr_g(\tilde{Y}))) = (p \cdot g, (r_g^*\tilde{w})(\tilde{Y}))\\ &= (p \cdot g, (r_g^*w_1)(\tilde{Y}) - (r_g^*w_0)(\tilde{Y})) = (p \cdot g, Ad(g^{-1})w_1(\tilde{Y}) - Ad(g^{-1})w_0(\tilde{Y}))\\ &= (p \cdot g, Ad(g^{-1}\tilde{w}(\tilde{Y})) \simeq (p, \tilde{w}(\tilde{Y}))\\ &\Rightarrow s \text{ is well-defined}\end{aligned}$$

From w_1 we get $w_1 - w_0 = \tilde{w}$ and from this we constructed s. Suppose we are given w_0 and an $s \in \Omega^1(M, Ad(P))$. We want to check that the combination gives a connection 1-form w_1 on P. For every $x \in M$ and $Y_x \in T_xM$, $s_x(Y_x) = [(p, Z)]$ for any $p \in \pi^{-1}(x)$ and some $Z \in \mathfrak{g}$. Define a $\mathfrak{g}$-valued 1-form $\tilde{w}$ on P as follows: for $A \in T_pP$ with $\pi(p) = x$ set $\tilde{w}(A) = Z$ if $s_x(D_p\pi(A)) = [(p, Z)]$.

$$\begin{aligned}(r_g^*\tilde{w})(A) &= \tilde{w}_{p \cdot g}(D_pr_g(A)) = Z = Ad(g^{-1})(Z)\\ [(p, Z)] &= s_x(D_p\pi(A)) = s_x(D_{p \cdot g}\pi(D_pr_g)(A)) = [(p \cdot g, Z')]\\ &= [(p, Ad(g)(Z'))]\\ &\Rightarrow Z = Ad(g)Z' \Rightarrow Z' = Ad(g^{-1})Z\\ \Rightarrow r_g^*\tilde{w} &= Ad(g^{-1})\tilde{w}\end{aligned}$$

Define $w_1 = w_0 + \tilde{w}$. Then:

$$r_g^* w_1 = r_g^* w_0 + r_g^* \tilde{w} = Ad(g^{-1})(w_0 + \tilde{w}) = Ad(g^{-1}) w_1$$

Let $Y \in \mathfrak{g}$ and Y^* the corresponding fundamental vector field. Then $w_1(Y^*) = \underbrace{w_0(Y^*)}_{=Y} + \underbrace{\tilde{w}(Y^*)}_{=0 \text{ since } Y^* \in ker D\pi} = Y$. So w_1 is a connecten 1-form. Fix w_0

$$w_1 \longrightarrow \tilde{w} \longrightarrow s$$

$$w_1 \longleftarrow \tilde{w} \longleftarrow s$$

□

Vorlesung 11: $\pi : P \to M$ a princ. G-bundle, w a connection 1-form on P. $\{U_i\}$ an open cover of M by trivializing sets for P.

$$\psi_i : \pi^{-1}(U_i) \to U_i \times G$$
$$s_i : U_i \to \pi^{-1}(U_i) \quad x \mapsto \psi_i^{-1}(x, e)$$

Define: $w_i = s_i^* w \in \Omega^1(U_i, \mathfrak{g})$. If $U_i \cap U_j \neq \emptyset$, we want to relate $w_i|_{U_i \cap U_j}$ to $w_j|_{U_i \cap U_j}$. The transition map

$$\psi_i \circ \psi_j^{-1} : (U_j \cap U_j) \times G \to (U_j \cap U_j) \times G$$

is of the form:

$$(x, g) \mapsto (x, \psi_{ij}(x) \cdot g)$$

where $\psi_{ij} : U_j \cap U_i \to G$ is a smooth map. On G we define

$$\theta \in \Omega^1(G, \mathfrak{g}) \quad \theta_g(X_g) = Y \text{ if } X_g = (Y^*)_g$$

Lemma 5.7. *If $x \in U_j \cap U_i$ and $X \in T_x M$, then*

$$w_j(X) = Ad(\psi_{ij}(x))^{-1} w_i(X) + \psi_{ij}^* \theta(X)$$

Proof.

$$\begin{aligned}
s_j(x) &= \psi_j^{-1}(x, e) = \psi_i^{-1} \circ \psi_i \circ \psi_j^{-1}(x, e) = \psi_i^{-1}(x, \psi_{ij}(x) \cdot e) \\
&= \psi_i^{-1}(x, e \cdot \psi_{ij}(x)) = \psi_i^{-1}(x, e) \cdot \psi_{ij}(x) = s_i(x) \cdot \psi_{ij}(x) = r_{\psi_{ij}(x)} s_i(x) \\
w_j(X) &= s_j^* w(X) = w(D_x s_j(X)) \\
&= w((D_{s_i(x)} r_{\psi_{ij}(x)} \circ D_x s_i + D l_{s_i(x)} \circ D_x \psi_{ij})(X)) \\
&= w(D_{s_i(x)} r_{\psi_{ij}(x)}(D_x s_i(X))) + w(D_x \psi_{ij}(X)) \\
&= (r^*_{\psi_{ij}(x)} w)(D_x s_i(X)) + \psi_{ij}^* \theta(X) \\
&= Ad(\psi_{ij}^{-1}(x)) w(D_x s_i(X)) + \psi_{ij}^* \theta(X) \\
&= Ad(\psi_{ij}(x))^{-1} w_i(X) + Y_{ij}^* \theta(X)
\end{aligned}$$

□

6 Parallel Transport

$\pi : P \to M$, w as before, $H = kerw$. At every point $p \in P$ we have $T_pP = V_p \oplus H_p$ with $H_p = kerw_p$. $D_p\pi : Z_pP \to T_{\pi(p)}M$ has kernel V_p.

$\Rightarrow D_p\pi|_{H_p} : H_p \to T_{\pi(p)}M$ is a isomorphism
$\Rightarrow$every $X \in T_{\pi(p)}M$ has a unique lift $X^* \in H_p \subset T_pP$under this isomorphism

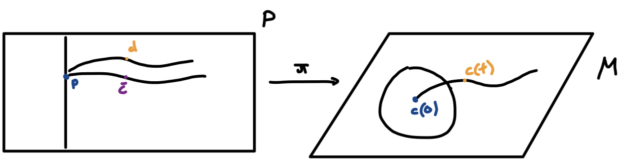
Every $X \in \mathfrak{X}(M)$ has a unique lift to a horizontal vector field $X^* \in \mathfrak{X}(P)$, i.e. $X^* \in \Gamma(H)$. Properties:

(1) $(fX)^* = \pi^*(f)X^*$

(2) $(X+Y)^* = X^* + Y^*$

(3) $([X,Y])^* = \mathfrak{H}[X^*, Y^*]$ where $\mathfrak{H} : TP \to H$ is the projection with kernel V.

[We will see that $[X^*, Y^*] - \mathfrak{H}[X^*, Y^*]$ if zero if and only if w is flat, i.e. H is integral]
Let $c : [0,1] \to M$ be a smooth curve. A **horizontal lift** $\bar{c}$ of c is a smooth curve $\bar{c} : [0,1] \to P$ with $\pi \circ \bar{c} = c$ and $\dot{\bar{c}}(t) \in H_{\bar{c}(t)} \quad \forall t \in [0,1]$.



Proposition 6.1. *For every $p \in \pi^{-1}(c(0))$ there is a unique horizontal lift $\bar{c}$ of c with $\bar{c}(0) = p$*

Proof. Given $p \in \pi^{-1}(c(0))$ choose an arbitrary smooth $d : [0,1] \to P$ with $d(0) = p$and $\pi \circ d = c$. Every other smooth lift of c is obtained from d by $\bar{c}(t) = d(t) \cdot g(t)$ for a smooth map $g : [0,1] \to G$. If $\bar{c}(0) = p = d(0)$, then $g(0) = e$. We went to find such a curve g in G so that $\dot{\bar{c}} \in H$.

$$\dot{\bar{c}}(t) = D_{d(t)} r_{g(t)}(\dot{d}(t)) + ((D_{g(t)} l_{g(t)^{-1}} \dot{g}(t))^* d(t) \cdot g(t))$$

$$
\begin{aligned}
0 = w(\dot{\bar{c}}(t)) &= (r^*_{g(t)}w)(\dot{d}(t)) + D_{g(t)}l_{g(t)^{-1}}\dot{g}(t) \\
&= Ad(g(t)^{-1})w(\dot{d}(t)) + D_{g(t)}l_{g(t)^{-1}}\dot{g}(t) \\
&= Dl_{g^{-1}(t)} \circ Dr_{g(t)}(w(\dot{d}(t))) + Dl^{-1}_{g(t)(\dot{g}(t))} \\
&= Dl_{g^{-1}(t)}(Dr_{g(t)}w(\dot{d}(t)) + \dot{g}(t)) \\
\Leftrightarrow & Dr_{g(t)} \underbrace{w(\dot{d}(t))}_{=X} + \dot{g}(t) = 0 \Leftrightarrow Dr_{g(t)}(X) = \dot{g}(t) \\
\Leftrightarrow & Dr^{-1}_{g(t)}(\dot{g}(t)) = X
\end{aligned}
$$

□

Lemma 6.2. *Let $X : [0,1] \to \mathfrak{g}$ be a smooth map. There is a unique smooth $g : [0,1] \to G$ with $g(0) = e$ and $D_{g(t)}r^{-1}_{g(t)}(\dot{g}(t)) = X(t) \quad \forall t$.*

Proof. Consider $G \times [0,1]$. On this we have a time-independent vector field Y given by $Y(g,t) = (X_g(t), \frac{\partial}{\partial t})$. This Y is complete with flow: $\varphi_t(e,0) = (g(t),t)$. This $g(t)$ solves the equation and is the only solution with $q(0) = e$. □

Vorlesung 12: $\pi : P \to M$ a princ. G-bundle with connection $H = kerw$. For every smoooth $c : [0,1] \to P$ with $\bar{c}(0) = p$ and $\pi \circ \bar{c} = c$ and $\dot{\bar{c}}(t) \in H_{\bar{c}(t)} \forall t$. This extends in the obvies way to a piecewise smooth c.

$$P_c : \pi^{-1}(c(0)) \to \pi^{-1}(c(1)) \quad \bar{c}(0) \mapsto \bar{c}(1)$$

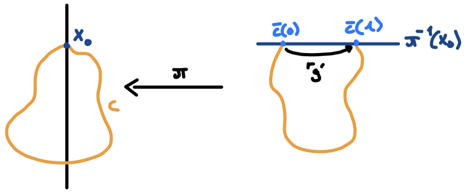
is the **parallel transport map** along c w.r.t. H. P_c does not change under oriantation-preserving re-parametrization of c. P_c is invertible with inverse $P_{\tilde{c}}$, where $\tilde{c}(t) = c(1-t)$. Fix a basepoint $x_0 \in M$ and consider c with $c(0) = x_0 = c(1)$. Then the P_c are self-maps of $\pi^{-1}(x_0)$.
Claim: " The maps P_c as s runs over all piecewise smooth c with $c(0) = c(1) = x_0$ form a group"

Proof. If $c(t) \equiv x_0$ then $P_c = Id$. If c_1, c_2 are two such loops, we concatenate them and get $P_{c_1 * c_2} = P_{c_2} \circ P_{c_1}$. This operation is associative! Inverse: $P_{\tilde{c}} = P_c^{-1}$. □

$\forall c \exists! g \in G$ such that $\bar{c}(1) = \bar{c}(0) \cdot g$ I.e. $P_c(\bar{c}(0)) = \bar{c}(0) \cdot g$
Claim: "g depends only on c (up to conjugation), and not on the choice of $\bar{c}(0) = p$"

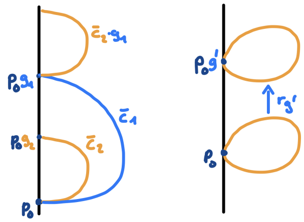
Proof. Suppose $c' : [0,1] \to P$ is the unique horizontal lift of c with $c'(0) = q$. Then $\exists! g' \in G : q = p \cdot g'$. Then $c'(t) = \bar{c}(t) \cdot g'$ by uniqueness of the horizontal lift. Then:

$$c'(1) = \bar{c}(1) \cdot g' = \bar{c}(0)gg' = (\bar{c}(0)g')(g')^{-1}gg' \quad c'(0) = \bar{c}(0) \cdot g'$$



Fix $p_0 \in \pi^{-1}(x_0)$. For all loops c staring and ending at x_0 we consider the loc. lifts $\bar{c}$ with starting point p_0. Then we have a $g \in G$ depending on c s.t. $\bar{c}(1) = \bar{c}(0) \cdot g = p_0 \cdot g$ □

Claim: " The map $\{P_c\} \to G$ $P_c \mapsto g$ is an injectie group homomorhpism.



Proof.

$$P_{c_1 * c_2}(p_0) = P_{c_2} \circ P_{c_2}(p_0) = P_{c_2}(p_0 \cdot g_1) = p_0 \cdot (g_2 g_1)$$

Suppose c is such that the corresponding $g \in G$ is e. Then $P_c(p_0) = p_0 \cdot e = p_0$. Consider $q = p_0 \cdot g'$:

$$P_c(q) = P_c(p_0) \cdot g' = p_0 g' = q \Rightarrow P_c = Id$$

□

Fix $p_0 \in \pi^{-1}(x_0)$.

Definition 6.3. $Hol(H, p_0) \subset G$ the **holonomy group** of H w.r.t. p_0 is the subgroup obtained as the image of $\{P_c\}$, where c ranges over all piece-wise smooth loops based at x_0. The **restricted holonomy group** $Hol_0(H, p_0)$ is the subgroup coming from contractinle loops c.

Proposition 6.4. *(1)* $Hol(H, p_0 \cdot g) = g^{-1} Hol(H, p_0) g$

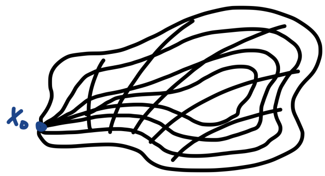
(2) If $q \in \pi^{-1}(x_0)$ *is obtained from* p_0 *by parallel transport, then* $Hol(H, q) = Hol(H, p_0)$

(3) *If q is obtained from p by parallel transport, then $Hol(H,q)$ and $Hol(H,p_0)$ are conjugate.*

Theorem 6.5. *The restricted holonomy group $H = Hol_0(H,p_0)$ is a connected Lie subgroup of G.*

Proof. By definition, H is a subgroup of G.

(1) For every $h \in H$ there is a path $\mu : [0,1] \to G$ s.t. $\mu(0) = e, \mu(1) = h$ and $\mu(t) \in H \forall t$



(2) The proof is completed by the following Prop.:

□

Proposition 6.6. *Let G be a Lie group, $H \subset G$ an abstract subgroup. If H has the property that every $h \in H$ can be connected to $e \in G$ by a piece-wise smooth curve in G with values in H, then H is a connected Lie subgroup.*

Proof. Define

$$\mathfrak{h} := \{\dot{c}(0) | c : [0,1] \to G \text{ piecewise smooth } c(0) = e \text{ and } c(t) \in H \forall t\}$$

It is clear that $\mathfrak{h} \subset T_eG \underset{ev-1,\cong}{\to} \mathfrak{g}$ is a linear subspace.[If ,c, d are such curves with $\dot{c}(0) = X, \dot{d}(0) = Y$, consider $\alpha : [0,1] \to G, t \mapsto c(t) \cdot d(t), \dot{\alpha}(0) = X + Y$]
Claim:"$\mathfrak{h}$ is a Lie subalgebra"

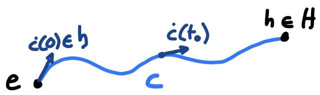
Proof. Take X, Y as above. Then consider:

$$\beta(t) = c(\sqrt{t})d(\sqrt{t})c(\sqrt{t})^{-1}d(\sqrt{t})^{-1} \quad \dot{\beta}(0) = [X,Y]$$

There is a connected Lie subgroup $K \subset G$ with Lie algebra $\mathfrak{h}$. We claim that K=H. $D_e l_g(\mathfrak{h}) = E_g$ defines a subbundle $E \subset TG$. Since $\mathfrak{h}$ is a Lie subalgebra, E is involutive, so integrable by Frobenius. K is the leaf of the corresponding foliation through e.

$$l_{c(t_0)^{-1}} : G \to G \quad c(t_0) \mapsto e \quad D_{c(t_0)} l_{c(t_0)^{-1}}(\dot{c}(t_0)) \in T_eG$$

is the velocity vector of $c^{-1}(t_0) \cdot c(t)$ ⇒this is in $\mathfrak{h}$ ⇒ by def. of E we conlude $\dot{c}(t_0) \in E_{c(t_0)} \Rightarrow c(t_9) \in K$ for all t_0, so $h \in K$. □



□

<u>*Vorlesung 13:*</u>

7 Curvature

$\pi : P \to M$ a princ. G-bundle, $H = kerw$ a connection.

Definition 7.1. The **curvature 2-form** Ω of H is:

$$\Omega(X,Y) := dw(\mathfrak{H}(X), \mathfrak{H}(Y))$$

where $\mathfrak{H} : TP \to H$ is the projection with kernel V.

$\Omega \in \Omega^2(P, \mathfrak{g})$

$$\begin{aligned}
(r_g^*\Omega)(X,Y) &= \Omega(Dr_g(X), Dr_g(Y)) \\
&= dw(\mathfrak{H}Dr_g(X), \mathfrak{H}Dr_g(Y)) \\
&= dw(Dr_g\mathfrak{H}(X), Dr_g\mathfrak{H}(Y)) \\
&= r_g^* dw(\mathfrak{H}(X), \mathfrak{H}(Y)) \\
&= d(r_g^*w)(\mathfrak{H}(X), \mathfrak{H}(Y)) \\
&= d(Ad(g^{-1}w)(\mathfrak{H}(X), \mathfrak{H}(Y)) \\
&= Ad(g^{-1})dw(\mathfrak{H}(X), \mathfrak{H}(Y)) = Ad(g^{-1})\Omega(X,Y)
\end{aligned}$$

(dont want Ω with $\mathfrak{H}$ in it)

Proposition 7.2. *(Structure equation)*

$$\Omega(X,Y) = dw(X,Y) + [w(X), w(Y)] \quad \forall X,Y \in TP$$

Proof. Both sides of the equation are bilinear and skew-symmetri. Therefore, it is enough to check the structure equation in the following 3 cases:

1. "$X, Y \in H$": $w(X) = 0 = w(Y)$ and $\Omega(X,Y) = dw(X,Y)$.

2. "$X, Y \in V$": W.l.og. A^*, B^* with $A^*_p, B^*_p,\ A, B \in \mathfrak{g}$ Then $w(X) = A$, $w(Y) = B$. $\Omega(X, Y) = 0$.

$$dw(X,Y) = L_{A^*}(\underbrace{w(B^*)}_{\equiv B}) - L_{B^*}(\underbrace{w(A^*)}_{\equiv A}) - w([A^*, B^*]) = 0 - 0 - [w(A^*), w(B^*)]$$
$$= -[A, B]$$

In this case the str. equation is true because both sides vanish.

3. "$X \in H, Y \in V$":

$$\Omega(X,Y) = dw(\mathfrak{H}(X), \mathfrak{H}(Y)) = dw(X, 0) = 0$$

Choose X^* horizontal v.f. with the given value X at $p \in P$ and A^* s.t. $A^*_p = Y, A \in \mathfrak{g}$.

$$dw(X,Y) = \underbrace{L_{X^*}(\underbrace{w(A^*)}_{\equiv A})}_{\equiv 0} - \underbrace{L_{A^*}(\underbrace{w(X^*)}_{\equiv 0})}_{\equiv 0} - \underbrace{w(\underbrace{[X^*, A^*]}_{\underset{7.3}{\in} H})}_{=0}$$
$$[w(X), w(Y)] = [w(X^*), w(A^*)] = [\underbrace{w(X^*)}_{\equiv 0}, A] = 0$$

Again in case 3. the str.eq.holds because both sides are $= 0$ by 7.3.

□

Lemma 7.3. *If X^* is hor. and A^* is fundamental, then $[X^*, A^*]$ is horizontal.*

Proof.

$$[X^*, A^*] = -[A^*, X^*] = -L_{A^*}X^*$$
$$= \frac{d}{dt} D\varphi_t(X^*)|_{t=0} \quad \text{where } \varphi_t \text{ is the flow of } A^*$$

Since A^* is fundamental, we have $\varphi_t(p) = r_{exp(tA)}(p)$. Since $X^* \in H$ and H is invariant under $r_g \forall g \in G, D\varphi_t(X^*)$ is again horizontal. $\Rightarrow L_{A^*}X^*$ is horizontal. □

Define $\mathfrak{D}$ an differential form on P by:

$$\mathfrak{D}\alpha(X_1, ..., X_{k+1}) = d\alpha(\mathfrak{H}(X_1), ..., \mathfrak{H}(X_{k+1})) \quad \text{if } deg\alpha = k$$

$\mathfrak{D}\alpha$ is called the **covariant derivative of** α(w.r.t. H). $\Omega = \mathfrak{D}w$

Proposition 7.4. *(Bianchi identity):* $D\Omega = 0$

Proof. Take $X, Y, Z \in H$. We have to prove that $d\Omega(X, Y, Z) = 0$.

$$\begin{aligned} dw(X,Y,Z) &= L_X(\Omega(Y,Z)) - L_Y(\Omega(Z,X)) + L_Z(\Omega(X,Y)) - \Omega([X,Y],Z) \\ &- \Omega([Z,X],Y) - \Omega([Y,Z],X) \\ &\underset{(*)}{=} L_X(dw(Y,Z)) + L_Y(dw(Z,X)) - L_Z(dw(X,Y)) \\ &- dw([X,Y],Z) - dw([Z,X],Y) - dw([Y,Z],X) \\ &= d(dw)(X,Y,Z) = 0 \end{aligned}$$

(*): $\Omega(X,Y) = dw(X,Y) + [w(X), \underbrace{w(Y)}_{=0}]$ ☐

Proposition 7.5. *Let $\pi : P \to M$ be a princ. G-bundle with a connection $H = ker(w)$. The following are equivalent:*

(1) $\Omega \equiv 0$

(2) H is involutive

(3) H is integrable

Proof. (2) $\Leftrightarrow$ (3) by 1.20.
$\Omega(X, Y) = 0$ if X or Y is vertical. If X, Y are both horizontal, then $\Omega(X, Y) = dw(X,Y) = \underbrace{L_X(w(Y))}_{=0} - \underbrace{L_Y(w(X))}_{=0} - w([X,Y])$

$$\begin{aligned} &\Omega \equiv 0 \Leftrightarrow \Omega(X,Y) = 0 \quad \forall X, Y \in H \Leftrightarrow w([X,Y]) = 0 \quad \forall X, Y \text{ hor.} \\ &\quad \Leftrightarrow [X,Y] \text{is hor. if } X, Y \text{ are} \Leftrightarrow H \text{ is involutive} \end{aligned}$$

☐

Definition 7.6. A connection is **flat** if $\Omega \equiv 0$

Let H be any connection (not. nec. flat). For $p \in P$ define

$$H(p_0) := \{q \in P | q \text{ can be obtained by parallel transport from } p_0\}$$

[If $\Omega \equiv 0$, then $H(p_0)$ is the leaf of p_0 in the hor. foliation integrating H.]
$H(p_0) \cap \pi^{-1}(\pi(p_0)) = p_0 \cdot Hol(H, p_0)$
$Hol_0(H, p_0)$ is a connected Lie subgroup of G. $Hol(H, p_0)$ has $Hol_0(H, p_0)$ as its connected component of e and it has countably many components, so $Hol(H, p_0)$ is a Lie group.

Proposition 7.7. *If M is connected, then $H(p_0)$ is a princ. $Hol(H, p_0)$-bundle.*

Proof. $\pi : H(p_0) \to M$ is the restriction of $\pi : P \to M$. We restrict the right G-action on P to the subgroup $Hol(H, p_0)$.

Claim: " This restricted action is simply transitive on the fibers of $\pi : H(p_0) \to M$":

Proof. It is enough to prove transitivity on the fiber of p_0. $Hol(H, p_0)$ is transitive on the fibre of p_0 by definition of $Hol(H, p_0)$. □

To prove that this simply transitive $Hol(H, p_0)$- action on $H(p_0)$ makes $H(p_0)$ into a princ. bundle we have to prove local triviality. Pick and open set U, s.t. $\pi(p_0) \in U$ and $\pi^{-1}(U)$ is trivial on P.

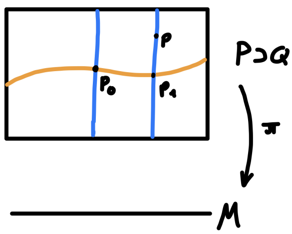
$$\psi : \pi^{-1}(U) \to U \times G \supset U \times Hol_0(H, p_0) \quad \text{(has smooth mfd. str.)}$$

as does : $U \times Hol(H, p_0)$. We can adjust ψ so that it sends $\pi^{-1}(U) \cap H(p_0)$ to $U \times Hol(H, p_0)$. This proves local triviality. □

Vorlesung 14:

$\pi : P \to M$ princ. G-bundle, H a connection. $H(p_0)$ the set of points in P that are parallel transports of p_0 w.r.t. H. If M is connected then $H(p_0)$ is a princ. $Hol(H, p_0)$-bundle. Suppose we have a reduction $f : Q \to P$ of the structure group of P to $G' \subset G$. Let H be a connection on Q.
Claim:"H extends uniquely to P"

Proof. For $p \notin Q \exists g \in G \exists p_1 \in Q : p = p_1 \cdot g$. Define $H_p := D_{p_1} r_g(H_{p_1}) \subset T_p P$. Well defined: p_2 another point in Q s.t. there is a $g' \in G$ with $p = p_2 \cdot g'$, then

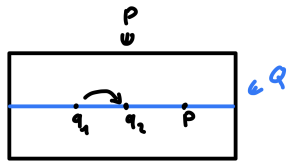


p_1, p_2 are both in Q and are in the same fiber of Q. $\exists! h \in G' : p_2 = p_1 \cdot h$.

$$p_1 \cdot g = p = p_2 \cdot g' = (p_1 \cdot h) \cdot g' = p_1 \cdot (hg') \Rightarrow g = h \cdot g' \Rightarrow g' = h^{-1} \cdot g$$

$$D_{p_2} r_{g'}(H_{p_2}) = D_{p_2 \cdot h^{-1}} r_g \circ D_{p_2} r_{h^{-1}}(H_{p_2}) = D_{p_1} r_g(H_{p_1})$$

so its well-defined and unique.
Claim:"This definition defines a connection on P"



Proof. The definition gives a horizontal subbundle on P. Check invariance under the right G-action: Suppose $q_1, q_2 \in P$ in the same fiber.

$$H_{q_1} = D_p r_{g_1}(H_p) \quad \text{if } p \cdot g_1 = p_1$$

$$H_{q_2} = D_p r_{g_2}(H_p) \quad \text{if } p \cdot g_2 = p_2$$

$$q_2 = p \cdot g_2 = (q_1 \cdot g_1^{-1})g_2 = q_1(g_1^{-1}g_2) \quad r_{g_1^{-1}q_2} = r_{g_2} \circ r_{g_1^{-1}}$$

$$D_{q_1} r_{g_1^{-1}g_2}(H_{q_1}) = D_p r_{g_2} \circ \underbrace{D_{q_1} r_{g_1^{-1}} \circ D_p r_{g_1}}_{Id}(H_p) = D_p r_{g_2}(H_p)$$

□

□

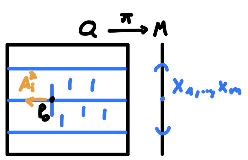
Definition 7.8. A connection H on P is **reducible** to a proper subgroup $G' \subset G$ if there is a connection $Q \to P$ of the structure of P to G' and a connection on Q whose extension to P is H.

Every connection H on P is reducible to $Hol(H, p_0)$.

Theorem 7.9. *(Ambrose-Singer):Let $\pi : P \to M$ be a princ. G-bundle with connection $H = ker w$. For $p_0 \in P$ define*

$$\mathfrak{g}' = \{\Omega_{p_0}(X, Y) | X, Y \in T_{p_0}P\} \subset \mathfrak{g}$$

Then $\mathfrak{g}'$ is the Lie algebra of $Hol(H, p_0)$.



Proof. We replace P by the holonomy subbundle $Q = H(p_0)$ with structure group $Hol(H, p_0)$. $\mathfrak{g}'$ is a sub-vectorspace of $\mathfrak{g}$, let $A_1, ..., A_k$ be a basis. Pick a local frame $X_1, ..., X_n$ for TM around $\pi(p_0)$. Lift the X_i to horizontal vector fields X_i^*. The $\{X_i^*\}$ form a local frame for H in a neighbourhood of p_0.

$$S := span\{A_1^*, ..., A_k^*, X_1^*, ..., X_n^*\} = span\{A_1^*, ..., A_k^*\} \oplus H$$

Claim:" S is integrable":

Proof. By 1.20 we only need to check that S is closed under $[,]$ (involutive). By bilinearity of $[,]$ we reduce to the following cases:

- $[A_i^*, A_j^*] = [A_i, A_j]^*$, so we need to prove that $\mathfrak{g}' \subset \mathfrak{g}$ is closed under $[,]$. More generally, we prove that for every $A \in \mathfrak{g}'$ and $B \in \mathfrak{g}$ we have $[B, A] \in \mathfrak{g}'$. $A = \Omega_{p_0}(X, Y)$ for some X, Y.

$$[B, A] = ad(B)(A) = \frac{d}{dt} Ad(exp(tB)(A))|_{t=0} \in \mathfrak{g}'$$

$$r_g^*\Omega = Ad(g^{-1})\Omega \quad Ad(g^{-1})\Omega_{p_0}(X, Y)) = (r_g^*\Omega)(X, Y) = \Omega_{p_0 g}(D_{p_0} r_g(X), D_{p_0} r_g(Y))$$
$$\Rightarrow Ad(g^{-1})A \in \mathfrak{g}' \quad \forall g \in G$$

- $[A_i^*, X_j^*]$ is horizontal if X_j^* is horizontal by a previous Lemma.

- $[X_j^*, X_i^*] = \underbrace{\mathfrak{H}[X_i^*, X_j^*]}_{\in H} + \underbrace{\mathfrak{V}[X_i^*, X_j^*]}_{=B^*}$ for some $B \in \mathfrak{g}$.

$$\begin{aligned}\Omega_{p_0}(X_i^*, X_j^*) = dw(X_i^*, X_j^*) &= L_{X_i^*} \underbrace{w(X_j^*)}_{=0} - L_{X_j^*} \underbrace{w(X_i^*)}_{=0} - w([X_i^*, X_j^*]) \\ &= -w(\mathfrak{V}[X_i^*, X_j^*]) = -B \in \mathfrak{g}' \\ &\Rightarrow \mathfrak{V}[X_i^*, X_j^*] \in S\end{aligned}$$

□

Let L be the leaf of the foliation defined by S passing through p_0. Every point in Q can be obtained from p_0 by parallel transport along some curve c.The horizontal lift $\bar{c}$ of c with starting point p_0 is everywhere tangent to $H \subset S$.$\Rightarrow$ $\bar{c}$ is contained in L, in particular its endpoint is in $L \Rightarrow L = Q$.

$$rdim(\mathfrak{g}') + dim(M) = k(S) = dim(L) = dim(Q) = dim(M) + dim(Hol(H, p_0))$$

$$\Rightarrow dim(Hol(H, p_0) = dim(\mathfrak{g}')$$

With $\mathfrak{g}' \subset L(Hol(H, p_0))$ we follow $\mathfrak{g}' = L(Hol(H, p_0))$ □

Vorlesung 15:

Remark 7.10. From last time: $\pi : P \to M$ a princ. G-bundle with connection $H = kerw$. $Q = H(p_0)$ the reduction to $Hol(H, p_0)$. The Lie algebra of $Hol(H, p_0)$ is the subspace of $\mathfrak{g}$ consiting of the values of Ω. We may work on Q.

$$\mathfrak{g}' = \{\Omega_p(X,Y) | \forall p \in Q; X, Y \in T_Q\} \subset L(Hol(H, p_0)$$

$$S = span\{A_1^*, ..., A_k^*\} \oplus H$$

If $X^*, Y^* \in H$, then $r_g^* \Omega = Ad(g^{-1})\Omega$ gives :

$$\underbrace{\Omega(Dr_g(X), Dr_g(Y))}_{\in \mathfrak{g}'} = (r_g^*\Omega)(X,Y) = Ad(g^{-1}) \underbrace{\Omega(X,Y)}_{\in \mathfrak{g}'}$$

$$\Omega(X,Y) = -w([X,Y])$$

$\mathfrak{g}' \subset L(Hol(H, p_0)$ is invariant under $Ad(g)$ if $g \in Hol(H, p_0) \Rightarrow \mathfrak{g}'$ is an ideal, in partic. it is a sub-Lie algebra.

Suppose $\pi_P : P \to M$ is a princ. G-bundle, and $\rho : G \to Gl(V)$ is a linear representation. Then $\pi_E : E \to M$ is the vector bundle associated to P via ρ:

$$E = P \underset{\rho}{\times} V = {}^{(P \times V)}/_{\simeq} \quad (p, v) \simeq (p \cdot g, \rho(g)^{-1} \cdot v) \quad \forall g \in G$$

Let $s \in \Gamma(E)$.Then $s(x) \in E_x = \pi_E^{-1}(x) \quad \forall x \in M$. Conisder $p \in P$ as a map $p : V \to E, v \mapsto [(p, v)]$ This parametrizes $E_{\pi_P(p))}$. Take $p \in \pi_p^{-1}(x)$, so that $\pi_P(p) = x$.Given s, there is a unique $v \in V$ such that $p(v) = s(x)$. This constructrion defines a map:

$$f : P \to V \quad f(p) = v$$

where v is the unique element of V with $p(v) = s(\pi_P(p))$ For $g \in G$, consider $f(p \cdot g) = w$, where

$$\begin{aligned} (p \cdot g)(w) &= s(x) = p(v) = (p \cdot g)(rho(g)^{-1} \cdot v) \quad x = \pi_P(p) \\ &\Rightarrow w = \rho(g^{-1})v \\ f(p \cdot g) &= \rho(g)^{-1} f(p) \quad \forall p, g \end{aligned}$$

Converse construction:Suppose $f : P \to V$ satisfies $f(p \cdot g) = \rho(g)^{-1} f(p)$. For $x \in M$ pick any $p \in \pi_P^{-1}(x)$ and define $s(x) = [(p, f(p)] \in E_x$. If $q \in \pi_p^{-1}(x)$, then $\exists! g \in G : q = p \cdot g$. Then:

$$(q, f(p)) = (p \cdot g, \rho(g)^{-1} f(p)) \simeq (p, f(p))$$

so s is well-defined. s smooth $\Leftrightarrow$ f smooth.

Proposition 7.11. *The curvature Ω of a connection $H = ker w$ on P can be interpreted as a section $F^w \in \Gamma(\bigwedge^2 T^*M \otimes Ad(P)) = \Omega^2(M; Ad(P))$*

Proof. Suppose $X, Y \in \mathfrak{X}(M)$: We want to define $F^w(X,Y)$ as a section of $Ad(P)$. Given H, X, Y have a unique hor.lifts $X^*, Y^* \in \Gamma(H)$. For a point $x \in M$ define:

$$F^w(X,Y)(x) := [(p, \Omega_p(X^*, Y^*))] \quad \text{where } p \in \pi_P^{-1}(x)$$

This defines a section by the previous contruction if $f = \Omega(X^*, Y^*) : P \to \mathfrak{g}$ satisfies $f(p \cdot g) = Ad(g)^{-1} f(p)$.

$$\Omega_{p\cdot g}(X^*, Y^*) = Ad(g^{-1})\Omega_p(X^*, Y^*)$$

Since X^*, Y^* are invariant under r_g, the LHS. is $(r_g^*\Omega)_p(X^*, Y^*)$. So the desired equation for f holds. Given F^w, we can reconstruct Ω as follows: given $Z, T \in \mathfrak{X}(P)$ we need to compute $\Omega(Z,T)$ through F^w. For $p \in P$:

1
$$\Omega_p(Z_p, T_p) = A \quad \text{if } F^w(D_p\pi(Z_p), D_p\pi(T_p)) = [(p, A)]$$

2 $\Omega(Z,T) = dw(\mathfrak{H}Z, \mathfrak{H}T)$

1,2 agree if one of Z and T is vertical. Both are bilinear. So it is enough to check that they agree on pairs of hor. vectors. If Z, T are horizontal, we may $Z = X^*, T = Y^*$ with $X, Y \in \mathfrak{X}(M)$. Then:

$$F^w(D\pi(Z), D\pi(T)) = [(p, \Omega_p(Z,T))]$$

□

8 Gauge transformations

Definition 8.1. A **global gauge transformation** on P is a diffeomorphism $\psi : P \to P$ s.t. $\psi(p \cdot g) = \psi(p) \cdot g$ and $\pi \circ \psi = \pi$

psi is an automorphism of the princ. bundle $\pi : P \to M$

Definition 8.2. $\mathfrak{G} = Aut(P)$ is the **gauge group** of P.

Proposition 8.3. *$\mathfrak{G}$ consists of the sections of $\pi_F : F \to M$ with fibre G associated to P by the conjugation action of G on itself.*

Proof. $F = {}^{(P \times G)}/_{\simeq}$ with $(p,h) \simeq (p \cdot g, g^{-1}hg$ Suppose $\phi \in \mathfrak{G}$. Then $\exists! u : P \to G$ such that $\phi(p) = p \cdot u(p)$.

$$\phi(p \cdot g) = (p \cdot g)u(p \cdot g) = \phi(p) \cdot g = p \cdot (u(p) \cdot g) = (p \cdot g) \cdot (g^{-1}u(p)g)$$
$$\Rightarrow u(p \cdot g) = g^{-1}u(p)g$$

Claim:"u defines a section of F"

Proof. For $x \in M$ define $s(x) = [(p, u(p))] \in F_x$. Well defines: $q = p \cdot g$: $(q, u(p)) = (p \cdot g, g^{-1}u(p)g) \simeq (p, u(p))$. Conversely ever section s of F gives a function $u : P \to G$ which defines $\phi : P \to P$ by $\phi(p) = p \cdot u(p)$ $\square$

$\square$

Remark 8.4. $\psi : Id_p = e \in \mathfrak{G}$ corresponds to $u : P \to G$ which is the constant map to $e \in G$. So F has a section. Since it is not usually trivia, it cannot be a princ. G-bundle.

Vorlesung 16:

$\mathfrak{G} = Aut(P)$ If $\phi \in \mathfrak{G}$, then ϕ acts on connections w by pullback : ϕ^*w. Let C be the space of connections on P. Define:

$$\mathfrak{G} \times C \to C \quad (\phi, w) \mapsto \phi^* w$$

$(\psi\phi)^*w = \phi^*\psi^*w$ This is a right G-action of C. Recall $Ad(P) = P \underset{Ad}{\times} \mathfrak{g} = {}^{(P \times \mathfrak{g})}/_{\simeq}$ where $(p, A) \simeq (p \cdot g, Ad(g^{-1})A) \quad \forall g \in G$. Define:

$$\phi([(p, A)]) = [(\phi(p), A)] = [(p, Ad(u(p))A)]$$

$$\phi([(p{\cdot}g, Ad(g^{-1})A)]) = [(\phi(p{\cdot}g), Ad(g^{-1})A)] = [\phi(p){\cdot}g, Ad(g^{-1})A)] = [(\phi(p), A)]$$

$$\mathfrak{G} \times Ad(P) \to Ad(P) \quad (\phi, [(p, A)]) \mapsto [(\phi(p), A)]$$

is a left action of $\mathfrak{G}$. Suppose $H = kerw$:

$$(\phi^* H)_p = (D_p\phi)^{-1} H_{\phi(p)}$$

$ker\phi^*w \ni X \Leftrightarrow 0 = (\phi^*w)(X_p) = w_{\phi(p)}(D_p\phi(X_p)) \Leftrightarrow D_p\phi(X_p) \in H_{\phi(p)} \Leftrightarrow X_p \in (D_p\phi)^{-1}(H_{\phi(p)})$ So $\phi^*H = ker(\phi^*w)$ if $H = kerw$.

$$\begin{array}{ccc} C & \xrightarrow{\phi^*} & C \\ \big\downarrow{\scriptstyle w \mapsto F^w} & & \big\downarrow{\scriptstyle \phi^*w \mapsto F^{\phi^*w}} \\ \Omega^2(M; Ad(P)) & \xrightarrow{?} & \Omega^2(M; Ad(P)) \end{array}$$

[Therefore take $X, Y \in T_xM$, $p \in \pi_P^{-1}(x)$, X^*, Y^* horizontal lifts of X, Y for

$\phi^* w$:

$$
\begin{aligned}
F^{\phi^* w}(X,Y) &= [(p, \Omega_p^{(\phi^* w)}(X^*, Y^*))] = (**) \\
\Omega_p^{(\phi^* w)}(X^*, Y^*) &= d\phi^* w(X^*, Y^*) + \underbrace{[\phi^* w(X^*), \phi^* w(Y^*)]}_{=0} \\
&= (\phi^* dw)(X^*, Y^*) = \phi^*(\Omega - [w,w])(X^*, Y^*) \\
&= \phi^*(X^*, Y^*) - [\underbrace{w(D\phi(X^*)}_{=0}, \underbrace{D\phi(Y^*))}_{=0}] \\
&= \phi^* \Omega(X^*, Y^*) \\
\phi^* \Omega_p^w(X^*, Y^*) &= \Omega^w_{\phi(p)}(D_p(X^*), D_p\phi(Y^*)) = (*) \quad \phi(p) = p \cdot u(p) \\
\Omega_{\phi(p)}(Z, T) &= \Omega_{p\cdot u(p)}(Z, T) = \Omega_{p \cdot u(p)}(D_p\phi(Z'), D_p\phi(T')) \\
&= r^*_{u(p)} \Omega(Z', T') = Ad(u(p))^{-1} \Omega(Z', T') \\
(*) &= Ad(u(p))^{-1} \Omega_p(X^*, Y^*) \\
(**) &= [(p, (\phi^* \Omega^w)_p(X^*, Y^*))] = [(p, Ad(u(p))^{-1} \Omega(X^*, Y^*))]
\end{aligned}
$$

X^*, Y^* are hor. lifts of X, Y w.r.t. $\phi^* w$, $\tilde{X}, \tilde{Y}$ hor. lifts w.r.t. w. $D\phi(X^*) = \tilde{X}, D\phi(Y^*) = \tilde{Y}$ so we get:

$$
\begin{aligned}
&\Rightarrow F^{\phi^* w}(X,Y) = [(p, Ad(u(p))^{-1} \Omega(X^*, Y^*))] \\
&\qquad F^w(X,Y) = [(p, \Omega(X^*, Y^*))] = [(\underbrace{p \cdot u(p)}_{=\phi(p)}, Ad(u(p))^{-1} \Omega(X^*, Y^*))] = \phi F^{\phi^* w}(X,Y)
\end{aligned}
$$

]
NEU: X^*, Y^* are hor. lifts of X, Y w.r.t. $\phi^* w$, $\tilde{X}, \tilde{Y}$ hor. lifts w.r.t. w. $D\phi(X^*) = \tilde{X}, D\phi(Y^*) = \tilde{Y}$:

$$
\begin{aligned}
F^{\phi^* w}(X,Y) &= [(p, (\phi^* \Omega^w)_p(X^*, Y^*))] = [(p, (\phi^* \Omega^w)_p(D\phi^{-1}_{\phi(p)}(\tilde{X}), D_{\phi(p)}\phi^{-1}(\tilde{Y}))] \\
&= [(p, \Omega^w_{\phi(p)}(D_p\phi \circ D\phi^{-1}_{\phi(p)}(\tilde{X}), D_p\phi \circ D\phi^{-1}_{\phi(p)}(\tilde{Y}))] \\
&= [(p, \Omega^w_{\phi(p)}(\tilde{X}, \tilde{Y}))] \\
F^w(X,Y) &:= [(p, \Omega^w_p(\tilde{X}, \tilde{Y}))] \underset{7.11}{=} [(p, Ad(u(p)) \Omega^w_{p \cdot u(p)}(\tilde{X}, \tilde{Y}))] \\
&= [(\phi(p), \Omega^w_{\phi(p)}(\tilde{X}, \tilde{Y}))] = \phi F^{\phi^* w}(X,Y)
\end{aligned}
$$

We get that the following diagram commutes:

$$
\begin{array}{ccc}
C & \xrightarrow{\quad \phi^* \quad} & C \\
\big\downarrow F & & \big\downarrow F \\
\Omega^2(M; Ad(P)) & \xrightarrow{\phi^{-1}} & \Omega^2(M; Ad(P))
\end{array}
$$

Proposition 8.5. *The curvature map $F : C \to \Omega^2(M; Ad(P))$ is $\mathfrak{G}$- equivariant for the right actions on C by pullback and on $\Omega^2(M; Ad(P))$ by acting with the inverse to turn the left action into a right action.*

Definition 8.6. Two connections are **gauge equivalent** if there is a $\phi \in \mathfrak{G}$ s.t. ϕ maps one to the other under pullback.

Corollary 8.7. *Flatness of connections is gauge invariant.*

Vorlesung 17:

C_P the space of connections on $\pi : P \to M$, $\mathfrak{G}_P$ the gauge group of P. $C_P^{flat} \subset C_P$ the subset of flat connections.

Theorem 8.8. $\coprod_P C_P^{flat}/\mathfrak{G}_p \underset{1:1}{\leftrightarrow} Hom(\pi_1(M), G)/_{conj.}$ *for fixed M, G, M is connected.*

Lemma 8.9. *If $H = ker w$ is a flat connection, then P_c depends only on the homotopy class of c (With fixed endpoints).*

Proof. H flat $\underset{A.S.Th.}{\Rightarrow}$ the Lie algebra of $Hol_0(H, p_0)$ is zero. $\Rightarrow Hol_0(H, p_0) = \{e\}$ this says that the holonomy is trivial for all null-homotopic paths. Suppose c_1, c_2 are closed loops based at x_0 that are homotopic with fixed endpoints. Then the composition $c_1 \cdot \bar{c_2}$ is null-homotopic with fixed endpoints, so

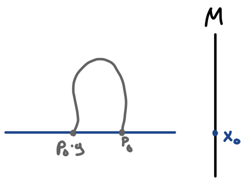
$$Id = P_{c_1 \cdot \bar{c_2}} = P_{\bar{c_2}} \circ P_{c_1} = P_{c_2}^{-1} \circ P_{c_1}$$

$\Rightarrow$ "$P_{c_1}(p_0) = p_0 \cdot g_1, P_{c_2}(p_0) = p_0 \cdot g_2 \Rightarrow g_1 = g_2$" □

Suppose H is flat. Fix $p_0 \in \pi^{-1}(x_0)$, define:

$$\rho : \pi_1(M, x_0) \to G \quad [c] \mapsto g^{-1} \quad \text{where } P_c(p_0) = p_0 \cdot g$$

This is well-defined by 8.9.



$\rho([c_1] * [c_2]) = g^{-1}$ if $P_{c_1 \cdot c_2}(p_0) = p_0 \cdot g = P_{c_2}(P_{c_1}(p_0) = P_{c_2}(p_0 \cdot g_1) = P_{c_2}(p_0) \cdot g_1 = p_0 \cdot (g_2 \cdot g_1)$

$\Rightarrow g = g_2 \cdot g_1, g^{-1} = g_1^{-1} \cdot g_2^{-1} \Rightarrow \rho$ is a homomorphism.
Replace p_0 has $p_1 = p_0 \cdot h$ for some $h \in G$. $P_c(p_0) = p_0 \cdot g_0 \Rightarrow P_c(p_1) = P_c(p_0 \cdot h) = P_c(p_0) \cdot h = p_0 \cdot g_0 \cdot h = p_1 \cdot \underbrace{h^{-1} \cdot g_0 \cdot h}_{g_1}$. $g_1^{-1} == h^{-1} g_0^{-1} h$. Conclusion: the conjugacy class of ρ is independent of the choice of $p_0 \in \pi^{-1}(x_0)$.

Lemma 8.10. *If H_1, H_2 are gauge equivalent flat connections on P, then there holonomy representations ρ_1, ρ_2 are conjugate.*

Proof. Let $\phi \in G$ be a gauge transformation with $D\phi(H_1) = H_2$. Pick $p_1 \in \pi^{-1}(x_0)$ to define ρ_1 w.r.t. H_1. Use $\phi(p_1) = p_2$ to define ρ_2 w.r.t. H_2. Fix a loop c beginning and ending at $x_0 \in M$.

$$P_c^{H_1}(p_1) = p_1 \cdot \rho([c])^{-1}$$

$p_2 \cdot \rho_2([c])^{-1} = P_c^{H_2}(p_2) = P_c^{D\phi(H_1)}(\phi(p_1)) = \phi(p_1 \cdot \rho_1([c])^{-1}) = \phi(p_1) \cdot \rho_1([c])^{-1} = p_2 \cdot \rho_1([c])^{-1} \Rightarrow \rho_1 = \rho_2$. M connected $x_0 \in M$. $\pi_1(M, x_0)$.

$$\pi : \tilde{M} := \{(x, [\gamma])\} \to M \quad (x, [\gamma]) \mapsto x$$

with $x \in M, \gamma$ a path from x_0 to x. $\tilde{M}$ the **universal covering** of M. $\pi_1(M, x_0)$ acts on $\tilde{M}$ on the left by:

$$\pi_1(M, x_0) \times \tilde{M} \to \tilde{M} \quad ([c], (x, [\gamma])) \mapsto (x, [c \cdot \gamma])$$

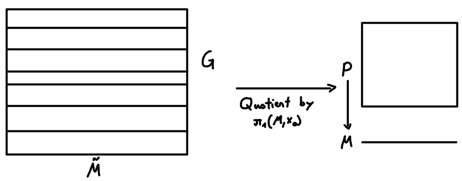
Let $\rho \in Hom(\pi_1(M, x_0), G)$. Define $P = \tilde{M} \underset{\rho}{\times} G = {}^{(\tilde{M} \times G)}/_{\simeq}$ where $((x, [\gamma]), g) \simeq ((x, [c \cdot \gamma]), \rho([c]) \cdot g)$.

CLaim:"P is a princ. G-bundle on M"

Proof. G acts on the right on P as follows

$$P \times G \to P \quad ([((x, [\gamma]), g)], h) \mapsto [((x, [\gamma]), g \cdot h)]$$

This is well-defined. The G- action is simply transitive on the fibers of $P \to M$. $\tilde{M} \times G \to \tilde{M}$ is a product princ. G-bundle. It has a flat connection whose foliation is the foliation by the $\tilde{M} \times \{g\}$ for fixed $g \in G$. The action of $\pi_1(M, x_0)$ on $\tilde{M} \times G$ preserves this foliation. Therefore, the foliation descends to P and gives a foliation complementary to the fibers of P.



The hor. foliation on P is invariant under the right G-action that makes P into a princ. G-bundle. So it defines a flat connection on P. □

□

Lemma 8.11. *If $\bar{\rho}$ is conjugate to ρ, then the flat connections constructed in this way from ρ resp. $\bar{\rho}$ are gauge equivalent.*

Proof. Suppose $\bar{\rho}(g) = \alpha \cdot \rho(g) \cdot \alpha^{-1} \quad \forall g \in G$ and a fixed $\alpha \in G$.

$$P_{\bar{\rho}} = {}^{(\tilde{M} \times G)}/_{\simeq} \quad ((x,[\gamma],g) \simeq ((x,[c\cdot\gamma]), \alpha\rho([c])\alpha^{-1}g)$$

Define :

$$\psi : P_{\bar{\rho}} \to P_\rho \quad [((x,[\gamma]),g)] \mapsto [((x,[\gamma]),\alpha^{-1}g)]$$

We claim that ψ is an isomorphism of $P_{\bar{\rho}}$ with P_ρ that sends the flat connection to the flat connection. If

$$\begin{array}{ccc} ((x,[\gamma]),g) \underset{w.r.t\bar{\rho}}{\simeq} & & ((x,[c\cdot\gamma]),\alpha\rho([c])\alpha^{-1}g) \\ \big\downarrow \psi & & \big\downarrow \psi' \\ ((x,[\gamma]),\alpha^{-1}g) \underset{w.r.t.\rho}{\simeq} & & ((x,[c\cdot\gamma]),\rho([c])\alpha^{-1}g) \end{array}$$

So ψ is well-defined and $P_\rho \circ \psi = P_{\bar{\rho}}$.

$$\psi([((x,[\gamma]),g)]\cdot h) = \psi([((x,[\gamma],g\cdot h)]) = [((x,[\gamma]),\alpha g h)] = \psi[((x,[\gamma]),g)]\cdot h$$

This ψ sends the hor. fol. of $P_{\bar{\rho}}$ to the hor. fol. of P_ρ □

Vorlesung 18:

Last time:

$$H \text{ flat} \to hol : \pi_1(M,x_0) \to G$$

$$^{(\tilde{M}\times G)}/_{\pi_1(M)} = P_s = \tilde{M} \underset{\rho}{\times} G \leftarrow \rho$$

proof of 8.8:

Proof. Start with $\rho \in Hom(\pi_1(M,x_0),G)$ and construct P_ρ with its flat connection H as above. Given $[c] \in \pi_1(M,x_0)$ and $p_0 \in \pi^{-1}(x_0) \subset P_\rho$ we let $\tilde{c}$ be the hor.lift (w.r.t. H) of c with starting point p_0. $\tilde{c}$ is a curve in the leaf of the foliation defined by H passing trough p_0. $hol([c]) = g^{-1}$ if the endpoint of $\tilde{c}$ is $p_0 \cdot g$. Points in P_ρ are of the form $[(x,[\gamma]),h)]$ where $((x,[\gamma]),g) \simeq ((x,[c/cdot\gamma]),\rho([c])\cdot h)$ for all $[c] \in \pi_1(M,x_0)$.

$$p_0 = [((x_0,[const.]),e)] \quad c : [0,1] \to M \text{ with } c(0) = c(1) = x_0$$

We lift c to $\tilde{\tilde{c}} : [0,1] \to \tilde{M}$ with starting point $(x_0,[const.])$. Then define $\tilde{c}(t) = [(\tilde{\tilde{c}}(t),e)] \in P_\rho$ This is horizontal for H and it is a lift of c with

$$\tilde{c}(0) = [\tilde{\tilde{c}}(0),e)] = [((x_0,[const.]),e)] = p_0$$

$$\tilde{c}(1) = [(\tilde{\tilde{c}}(1),e)] = [((x_0,[c]),e)] = [((x_0,e),\rho([c])^{-1})] = p_0\cdot\rho([c])^{-1}$$

$$\Rightarrow \text{hol rep. of } H \text{ is } \rho$$

Conversely: [Idea: start with a flat connection on some $\pi_P : P \to M$. Construct its holonomy rep. hol. Using hol, construct P_{hol} with its flat connection $\bar{H}$. We need to prove that P and P_{hol} are isomorphic in such a way that H corresponds to $\bar{H}$ under the isomorphism.]
Fix $p_0 \in \pi_P^{-1}(x_0)$ and let $H(p_0)$ be the holonomy bundle of p_0 w.r.t H. $H(p_0)$ is the leaf through p_0 of the foliation defined by H. Then $H(p_0) = \tilde{M}/\Gamma$, $\Gamma \subset \pi_1(M, x_0)$. We write $p \in H(p_0)$ as $[x]$ where $x \in \tilde{M}$. $P_{hol} = (\tilde{M} \times G)/\pi_1(M, x_0)$
Define:

$$\phi : H(p_0) \to P_{hol} \quad [x] \mapsto [x, e]$$

Suppose $[y] = [x]$. Then $\exists [c] \in \Gamma \subset \pi_1(M, x_0) : y = [c] \cdot x$

$$[(y, e)] = [\underbrace{([c] \cdot x, e)}_{\in \tilde{M} \times G}] = [(x, hol([c])^{-1})] = [(x, e)] \quad \text{since } \Gamma = ker(hol)$$

ϕ is well-defined and smooth. Let $q \in P$ there is a $g \in G$ s.t.: $q \cdot g \in H(p_0)$. Define: $\phi(q) = \phi(q \cdot g) \cdot g^{-1}$ This is well-defined, smooth and invertible. ϕ is an isomorphism from P to P_{hol} s.t. $D\phi(H) = \bar{H}$. □

M compact $\Rightarrow \pi_1(M, x_0) =< g_1, ..., g_a | r_1, ..., r_b >$ is a finitely presented group. $Hom(\pi_1(M, x_0)) = S \subset \underbrace{G \times .... \times G}_{a-times}$ where S consists of the a-tuples $(g_1, ..., g_a)$ satisfying $r_1, ..., r_b$.

9 The Yang-Mills functional

$\pi : P \to M$ a princ. G-bundle over a connected mfd. M. Assume M is compact, oriented and equipped with a Riemannian metric. The orientation together with the metric defines a volume form $dvol$ on M by the requirement that $dvol(e_1, ..., e_n) = +1$ on any pos. oriented orthonormal basis $e_1, ..., e_n$.

$$\int_P ||\Omega||^2 = ?$$

For w a connection on P consider $F^w \in \Gamma(\bigwedge T^*M \otimes Ad(P))$. Choose a pos.definite $Ad(G)$-invariant scalar product on $\mathfrak{g}$.This defines a fibre-wise scalar product or metric on $Ad(P)$.

Definition 9.1.

$$\mathfrak{YM}(w) := \int_M ||F^w||^2 dvol \quad \mathfrak{YM} : C_p \to \mathbb{R}$$

Yang-Mills-functional

Lemma 9.2. *For any $\phi \in \mathfrak{G}_p$, we have $\mathfrak{YM}(\phi^* w) = \mathfrak{JM}(w)$, so $\mathfrak{YM}$ is gauge invariant. So $\mathfrak{YM}$ descends to ${}^{C_P}/_{\mathfrak{G}_P}$*

<u>*Vorlesung 19:*</u>

Proposition 9.3. *Let ψ be a one-form on P with values in $\mathfrak{g}$ s.t. $r_g^*\psi = Ad(g)^{-1}\psi$ and such that $\psi(X) = 0$ if X is vertical. Then*

$$D\psi(X,Y) = d\psi(X,Y) + [w(X), \psi(Y)] + [\psi(X), w(Y)]$$

with D defined by $D\psi(X,Y) = d\psi(\mathfrak{H}X, \mathfrak{H}Y)$ where $\mathfrak{H}$ is horizontal projektion w.r.t. w.

Proof. Both sides are bilinear and skew-symetric. So it suffices to check the following 3 cases:

Case 1: $\mathfrak{H}X = X, \mathfrak{H}Y = Y$ and so $D\psi(X,Y) = d\psi(X,Y)$ by definition; moreover $w(X) = 0, w(Y) = 0$.

Case 2: $X, Y \in V \Rightarrow \psi(X) = 0, \psi(X) = 0$ and also $\mathfrak{H}X = 0 = \mathfrak{H}Y$, so $D\psi(X,Y) = 0$. We may assume $X = A^*, Y = B^*$. Then

$$d\psi(X,Y) = L_X(\psi(Y)) - L_Y(\psi(X)) - \psi([X,Y]) = L_{A^*}(\psi(0)) - L_{B^*}(0) - \psi(\underbrace{[A,B]^*}_{=0}) = 0$$

Case 3: We may take $X = A^*$, Y be the value of a G-invariant hor.vector field Y^*. $x \in H \Rightarrow D\psi(X,Y) = 0, \psi(X) = 0$

$$d\psi(X,Y) = L_{A^*}(\psi(Y^*)) - L_{Y^*}(\underbrace{\psi(A^*)}_{=0}) - \psi(\underbrace{[A^*, Y^*]}_{=0})$$

$$\begin{aligned} L_{A^*}(\psi(Y^*)) &= \frac{d}{dt}\psi(Y^*)_{\psi_t(p)}|_{t=0} = \frac{d}{dt}(r_{g_t}^*\psi)(Y^*)|_{t=0} \\ &= \frac{d}{dt}Ad(g_t)^{-1}\psi(Y^*)|_{t=0} = [-A, \psi(Y^*)] \end{aligned}$$

$$= -[w(A^*), \psi(Y^*)]$$

ϕ_t the flow of A^* $\quad \phi_t(p) = p \cdot exp(tA) =: p \cdot g_t$

□

Let w_0 be a connection 1-form on P and ψ as in 9.3. Then $w_t = w_0 + t\psi, t \in [0,1]$ is a smoothly varying family of connection 1-forms.

$$\begin{aligned} \Omega_t = D_t w_t &\underset{str.eq.}{=} dw_t + [w_t, w_t] = d(w_0 + t\psi) + [w_0 + t\psi, w_0 + t\psi] \\ &= dw_0 + [w_0, w_0] + t(d\psi + [w_0, \psi] + [\psi, w_0]) + t^2[\psi, \psi] \\ &= \Omega_0 + tD_0\psi + t^2 ... \qquad w_t \text{ with } \dot{w}_0 = \psi \end{aligned}$$

$$\frac{d}{dt}\Omega_t|_{t=0} = D_0\psi = D_0((\dot{w})|_{t=0})$$

As ψ as in 9.3 can be thought of as $\psi \in \Omega^1(M; Ad(P))$

$$D_0 : \Omega^1(M; Ad(P) \to \Omega^2(M; Ad(P)) \quad \psi \mapsto D_0\psi$$

Having chosen a Riemannian metric on M and a $Ad(G)$-invariant scalar product on $\mathfrak{g}$, we have metrics on the domain and tangent of D_0. For $\psi, \phi \in \Omega^i(M, Ad(P))$ we put :

$$< \psi, \phi >_{L^2} := \int_M < \psi, \phi > dvol \qquad M \text{ cpt,orientable}$$

Let

$$D_0^* : \Omega^2(M; Ad(P)) \to \Omega^1(M; Ad(P))$$

be the formal adjoint of D_0. Let w_t be a smooth family of connection 1-forms as above.

$$\mathfrak{YM}(w_t) = \int_M ||F^{w_t}||^2 dvol \quad F^{w_t} \text{ corresponds to } \Omega_t$$

$$\begin{aligned} \frac{d}{dt}\mathfrak{YM}(w_t) &= \frac{d}{dt}\int_M < F^{w_t}, F^{w_t} > dvol = 2\int_M < F^{w_t}, \frac{d}{dt}F^{w_t} > dvol|_{t=0} \\ &= 2\int_M < F^{w_0}, D_0\psi > dvol = 2\int_M < D_0^*F^{w_0}, \psi > dvol \end{aligned}$$

Proposition 9.4. *w_0 is a critical point of $\mathfrak{YM} \Leftrightarrow D_0^*F^{w_0} = 0$*

Remark 9.5. $D_0F^{w_0} = 0$ always by the Bianchi identity.

Remark 9.6. The **YM-equation** $D_0^*F^{w_0}$ is a 2^{nd} order PDE for w_0.

Example 9.7. *$G = S^1 = SO(2) = U(1), \mathfrak{g} = (i)\mathbb{R}$ is Abelian $\Rightarrow$ Ad is the trivial rep.$\Rightarrow$ $Ad(P)$ is trivial of rank 1 over M. For any princ. S^1-bundle $\pi : P \to M$ the space of connections is an affine space for $\Omega^1(M)$. The curvature of any connection on P is just a 2-form on $M.F^w \in \Omega^2(M)$ s.t.*

$$\pi^*F = \Omega = dw + \underbrace{[w, w]}_{=0}$$

$dF^w = 0$ by the bianchi identity. [$0 = d\Omega = d\pi^*F^w = \pi^*(dF^w) \Rightarrow dF^w = 0$ becouse π^* is injective] Since dF^w is closed, it defines $[F^w] \in H^2_{dR}(M) = \frac{ker(d:\Omega^2\to\Omega^3)}{Im(d:\Omega^1\to\Omega^2)}$.

Lemma 9.8. *$[F^w]$ is independent of the choice of w.*

Proof. Every connection 1-form on P is of the form $w + \pi^*\psi$ for some 1-form on M. Then the curvature of $w + \pi^*\psi$ is $dw + d\pi^*\psi = dw + \pi^* d\psi$:

$$\Rightarrow F^{w+\pi^*\psi} = F^w + d\psi \Rightarrow F^{w+\pi^*\psi} \text{ and } F^w \text{ have difference an exact form}$$

□

Definition 9.9. $\mathfrak{e}(P) = \frac{1}{2\pi}[F^w] \in H^2_{dR}(M)$ is the **Euler class** or **first Chern class** of P.

Let $\mathfrak{D}_P \subset ker(d : \Omega^2(M) \to \Omega^3(M))$ be the subset of the closed forms consisting of those representing $2\pi \cdot \mathfrak{e}(P)$

Lemma 9.10. *Every form in $\mathfrak{D}_P$ is the curvature of a suitable connection on P.*

Proof. Let $\alpha \in \mathfrak{D}_P$, and w a connection on P. Then $[F^w] = [\alpha] \in H^2_{dR}(M) \Rightarrow \alpha = F^w + d\psi = F^{w+\pi^*\psi}$ □

Vorlesung 20:

$\pi : P \to M$ a princ. S^1-bundle. $\mathfrak{e}(P) \in H^2_{dR}(M)$. $\mathfrak{D}_P$ the space of closed 2-forms representing $2\pi\mathfrak{e}(P)$.

$$C_P \to \Omega^2(M) \quad w \mapsto F^w \quad \text{ is a surjection onto } \mathfrak{D}_P \subset \Omega^2(M)$$

$F^{w+\pi^*\psi} = F^w + d\psi$ from that we get:

$$F^{w+\pi^*\psi} = F^w \Leftrightarrow d\psi = 0$$

This means "$w \to F^w$" has fibre the space of closed 1-forms. Let $\mathfrak{G}_P$ be the gauge group of P:

$$\mathfrak{G}_P = \{\psi : P \to P\} = \{u : M \to S^1 | u(p \cdot g) = \underbrace{g^{-1}u(p)g}_{u(p),\text{becouse } S^1 \text{ is abelian}} \quad \forall g \in S^1\}$$

$$= \{\bar{u} : M \to S^1\}$$

$$\begin{array}{ccc} P & \xrightarrow{u} & S^1 \\ \big\downarrow \pi & \nearrow \bar{u} & \\ M & & \end{array}$$

$\psi \in \mathfrak{G}_P$ acts on C_P by $\psi^* w = Ad(u)^{-1}w + u^*\theta = w + u^*\theta$.

$$F^{\psi^* w} = F^w + d\bar{u}^*\theta = F^w + \bar{u}^* \underbrace{d\theta}_{=0} = F^w$$

This says that the curvature map "$w \mapsto F^w$" is constant on each $\mathfrak{G}_P$-orbit.

$$\begin{array}{ccc} & & \mathbb{R} \\ & \overset{\tilde{u}}{\nearrow} & \downarrow{\scriptstyle exp} \\ M & \xrightarrow{\bar{u}} & S^1 \end{array}$$

$\bar{u}$ lifts to $\tilde{u} \Leftrightarrow \bar{u}$ is null-homotopic. $[M, S^1] = H^1(M; \mathbb{Z})$ the set of homotopy classes of maps $M \to S^1$.

$[M, S^1] \underset{1:1}{\to} H^1(M, \mathbb{Z}) \quad [f] \mapsto f^*\alpha \quad$ where α is the generator of $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$

putting this together we get the map:

$$1 \to \mathfrak{G}_0 \to \mathfrak{G}_P \to H^1(M; \mathbb{Z}) \to 0$$
$$\bar{u} \mapsto [\bar{u}]$$

$\mathfrak{G}_0$ is the space of null-homotopic $\bar{u}$, which is the connected component of $e \in \mathfrak{G}_P$. Suppose $\bar{u} \in \mathfrak{G}_0$, so it lifts to $\tilde{u} = f : M \to \mathbb{R}$. $\bar{u}(x) = e^{2\pi i f(x)}$. Let ψ be the gauge transformation defined by $\bar{u}$ for $u = \pi^*\bar{u}$. Then $\psi^* w = w + u^*\theta \Rightarrow \psi^* w - w = \bar{u}^*\theta$(as a one form on the base) Note $\bar{u} = exp \circ \tilde{u}$ and $\bar{u}^*(\theta) = \tilde{u}^* \circ exp^*(\theta) = \tilde{u}^*(dt) = f^*dt = df$

$$curv. : C_P \to \mathfrak{D}_P \quad curv^{-1}(F) = ker(d : \Omega^1(M) \to \Omega^2(M))$$

$\mathfrak{G}_0$ acts on C_P preserving the fibres of curv. so in particular $\mathfrak{G}_0$ acts on $curv^{-1}(F)$.

$$curv^{-1}(F)/\mathfrak{G}_0 = \frac{ker(d : \Omega^1(M) \to \Omega^2(M))}{Im(d : \Omega^0(M) \to \Omega^1(M))} = H^1_{dR}(M)$$

$\mathfrak{G}$ acts on C_P and is constant on the fibers of the map $curv. : C_P \to \mathfrak{D}_P$. So the map descends to $\mathfrak{B} := (C_P/\mathfrak{G}) \underset{c}{\to} \mathfrak{D}_P$. Here $c^{-1}(F) \cong H^1_{dR}(M)/H^1(M; \mathbb{Z}) = T^{b_1(M)}$. If $b_1(M) = 0$, this is a point.

Proposition 9.11. *The gauge equivalence classes of connections on P having a prescribed curvature 2-form F are parametrized by $T^{b_1(M)}$.*

If P admits a flat connection, then $\mathfrak{c}(P) = 0$. Take P to be the trivial bundle. The space of gauge equivalence classes of flat connections on P is parametrized by $T^{b_1(M)}$.

$$\begin{aligned} Hom(\pi_1(M, x_0), S^1)/conj. &= Hom(\pi_1(M, x_0), S^1) = Hom(H_1(M; \mathbb{Z}), S^1) \\ &= \underbrace{S^1 \times .... \times S^1}_{rk H_1(M;\mathbb{Z}) = b_1(M)} \end{aligned}$$

For arbitrary P, the **YM-connections** are the ones satisfying $D_0^* F w_0 = 0$. If $G = S^1$, then $D_0 = d$ is the exterior derivative. For any connection w we have $dF^w = 0$. The YM-equation for w is $d^* F^w = 0$.

$$\triangle = dd^* + d^*d \quad \text{the } \textbf{laplace operator}$$

Definition 9.12. A form α is **harmonic** is $\triangle\alpha = 0$

Lemma 9.13. *M compact, oriented, Riemannian:*

$$\triangle\alpha \Leftrightarrow d\alpha = 0 = d^*\alpha$$

Proof. "$\Leftrightarrow$": clear.
"$\Rightarrow$": Suppiose $\triangle\alpha = 0$. Then:

$$\begin{aligned} 0 &= \int_M < \triangle\alpha, \alpha > dvol = \int_M < dd^*\alpha, \alpha > dvol + \int_M < d^*d\alpha, \alpha > dvol \\ &= \int_M |d^*\alpha|^2 dvol + \int_M |d\alpha|^2 dvol \\ &\Rightarrow d\alpha = 0, d^*\alpha = 0 \end{aligned}$$

□

Theorem 9.14. *(Hodge Theorem) M cpt., orientable, Riemannian. Every* $[\alpha] \in H^k_{dR}(M)$ *has a unique harmonic representative.*

Proposition 9.15. *For every princ.* S^1*-bundle over* M*, a connection* W *is a YM-connection if and only if* $\frac{1}{2\pi}F^w$ *is the unique harmonic representative of* $\mathfrak{e}(P) \in H^2_{dR}(M)$*. The gauge equivalence classes of YM-connections on* P *are parametrized by* $T^{b_1(M)}$

Vorlesung 21:
M a cpt. oriented Riemannian manifold $<,>$ dvol is the volume form defined by $dvol(e_1, .., e_n) = 1$ for any oriented o.n. basis $e_1, ..., e_n$. $<,>$ on TM induces a metric on all $\bigwedge^{n-k} T^*M$.

Definition 9.16.

$$* : \bigwedge^k T^*M \to \bigwedge^{n-k} T^*M$$

the **hodge star** is defined by $\alpha \wedge *\beta = < \alpha, \beta > dvol \quad \forall \alpha, \beta$.

V an orient. n-dim $\mathbb{R}-$vector space with scalar product $<,>_0$ the scalar product defines an isomorphism:

$$f_0 : V \to V^* \quad v \mapsto (w \mapsto < v, w >_0)$$

The scalar product on V^* is defined by requiring that f_0 is an isometry.

Let $<,>_1 := \lambda^2 <,>_0$ with $\lambda > 0$. Suppose $e_1, .., e_m$ is o.n. w.r.to $<,>_0$

$$< e_j, e_i >_1 = \lambda^2 \underbrace{< e_j, e_i >_0}_{=1}$$

$$\frac{1}{\lambda} e_1, ..., \frac{1}{\lambda} e_n \text{a o.n.b. w.r.t. } <,>_1$$

$$f_1 : V \to V^* \quad v \mapsto (w \mapsto < v, w >_1)$$

$$f_1(v)(w) = \lambda^2 < v, w >_0 = \lambda^2 f_0(v)(w) \quad \forall w \in V$$

$$\Rightarrow f_1 = \lambda^2 f_0$$

On V^* we have :

$$1 =< f_1(\frac{1}{\lambda} e_i), f_1(\frac{1}{\lambda} e_i) >_1 = \frac{1}{\lambda^2} < f_1(e_i), f_1(e_i) >_1$$

$$= \frac{1}{\lambda^2} < \lambda^2 f_o(e_i), \lambda^2 f_0(e_i) >_1 = \lambda^2 < f_0(e_i), f_0(e_i) >_1$$

$$\Rightarrow < f_0(e_i), f_0(e_i) >_1 = \frac{1}{\lambda^2} < f_0(e_i), f_0(e_i) >_0$$

If $\alpha_1, ..., \alpha_n$ are o.n. w.r.t. $<,>_0$ on V^*, then $\alpha_{i_1} \wedge ... \wedge \alpha_{i_k}$ are o.n. in $\bigwedge^k V^*$. This defines $<,>_0$ for $k \geq 2$.
$dvol_1 = c \cdot dvol_0$ with $c > 0$.

$$1 = dvol_1(\frac{1}{\lambda} e_1, ..., \frac{1}{\lambda} e_n) = \frac{1}{\lambda^n} dvol_1(e_1, ..., e_n)$$

$$= \frac{1}{\lambda^n} c \cdot \underbrace{dvol_o(e_1, .., e_n)}_{=1}$$

On 1-forms $< \alpha, \beta >_1 = \frac{1}{\lambda^2} <, \beta >_0$. If $\alpha_1, ..., \alpha_n$ are o.n.w.r.t. $<,>_0$, then $\lambda\alpha_1, ..., \lambda\alpha_n$ are o.n.w.r.t.$<,>_1$.

$$\Rightarrow \lambda\alpha_{i_1} \wedge ... \wedge \lambda\alpha_{i_k} \text{are o.n.w.r.t. } <,>_1 \text{ on } \bigwedge^k V^*$$

$$= \lambda^k (\alpha_{i_1} \wedge ... \wedge \alpha_{i_k} \Rightarrow < \alpha, \beta >_1 = \frac{1}{\lambda^{2k}} < \alpha, \beta >_0 \; on \bigwedge^k V^*$$

Let $*_0$ and $*_1$ be the hodge star operators defined by $<,>_0$ resp. $<,>_1$, using the same orientation of V. Then:

$$\alpha \wedge *_1 \beta =< \alpha, \beta >_1 dvol_1 = \frac{1}{\lambda^{2k}} < \alpha, \beta >_0 \cdot \lambda^n dvol_0$$

$$= \lambda^{n-2k} < \alpha, \beta >_0 dvol_0 = \lambda^{n-2k} \alpha \wedge *_0 \beta$$

$$\Rightarrow \text{On } \bigwedge^k V^* \text{ we have } *_1 = \lambda^{n-2k} \cdot *_0$$

Proposition 9.17. *If $dim M = n$ is even and $k = \frac{1}{2}n$, then the Hodge star on $\Omega^k(M)$ is conformally invariant.*

Remark 9.18. $<,>_1 = f^2 <,>_0$ with $f > 0$, f does not have to be constant.

M cpt., oriented, without boundary:

$$< \alpha, \beta >_{L^2} := \int_M < \alpha, \beta > dvol$$

Lemma 9.19. *The formal adjoint of $d : \Omega^k(M) \to \Omega^{k+1}(M)$ w.r.t. $<,>_{L^2}$ is :*

$$d^* = (-1)^{1+k+n(n-k)} * d*$$

Proof. Take $\alpha \in \Omega^k(M), \beta \in \Omega^{k+1}(M)$.

$$\begin{aligned}
0 &\underset{stokes}{=} \int_M d(\alpha \wedge *\beta) = \int_M (d\alpha \wedge *\beta + (-1)^k \alpha \wedge d(*\beta)) \\
&= \int_M < d\alpha, \beta > dvol + (-1)^k \int_M \alpha \wedge * * d(*\beta) \cdot (-1)^{n(n-k)} \\
&\Rightarrow < d\alpha, \beta >_{L^2} = (-1)^{1+k+n(n-k)} \int_M \alpha \wedge *(*d * \beta) \\
&= (-1)^{1+k+n(n-k)} < \alpha, *d * \beta >_{L^2} \\
&\Rightarrow d^* = (-1)^{1+k+n(n-k)} * d*
\end{aligned}$$

□

Exercise: $** = (-1)^{n(n-k)}$ on $\bigwedge^k V^*$
$d^*\alpha = 0 \Leftrightarrow d(*\alpha) = 0$

Corollary 9.20. *Harmonic 2-forms on a 4-mfd are conformally invariant.*

Take $dim M = 4$. Fix $<,>$ and a orientation.

$$H^2_{dR}(M) = \mathfrak{H}(M) = \{\alpha \in \Omega^2(M) | \triangle\alpha = 0\} = \{\alpha \in \Omega^2(M) | d\alpha = 0 = d(*\alpha)\}$$

$$Q_M : H^2_{dR}(M) \times H^2_{dR}(M) \to \mathbb{R} \quad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta$$

the intersection/cup product paring of M.

- well-defined (by stokes)
- symmetric
- non-degenerate, i.e. $\forall [\alpha] \in H^2_{dR}(M) \exists [\beta] s.t. \int_M \alpha \wedge \beta \neq 0$

$* : \Omega^2(M) \to \Omega^2(M)$ maps $\mathfrak{H}^2(M) \subset \Omega^2(M)$ to itself. $** = Id$, so we can slplit $\mathfrak{H}^2(M) = \mathfrak{H}^2{}_+ \oplus \mathfrak{H}^2_-$ into the (+1/-1) eigenspaces of $*$.

Definition 9.21. A 2 form α is called **SD** if $*\alpha = \alpha$ and is called **ASD** if $*\alpha = -\alpha$

Suppose $0 \neq \alpha \in \mathfrak{H}^2{}_+$. Then:

$$Q_M([\alpha],[\alpha]) = \int_M \alpha \wedge \alpha = \int_M \alpha \wedge *\alpha = ||\alpha||^2_{L_2} > 0$$

$\Rightarrow Q_M$ is pos. definite on $\mathfrak{H}^2_+ \subset H^2_{dR}(M)$. If $0 \neq \alpha \in \mathfrak{H}^2_-$, then

$$Q_M([\alpha],[\alpha]) = \int_M \alpha \wedge \alpha = -\int_M \alpha \wedge *\alpha = -||\alpha||^2_{L^2} < 0$$

$\Rightarrow Q_M$ is neg. definite on $\mathfrak{H}^2_-$.
Take $\alpha \in \mathfrak{H}^2_+, \beta \in \mathfrak{H}^2_-$. Then:

$$\begin{aligned} Q_M([\alpha][\beta]) &= \int_M \alpha \wedge \beta = -\int \alpha \wedge *\beta = -\int < \alpha, \beta > dvol \\ &= 0 \quad \text{bec. } \alpha, \beta \text{ are positive orthogonal} \\ < \alpha, \beta > dvol &= \alpha \wedge *\beta = -\alpha \wedge \beta = -(*\alpha) \wedge \beta = -\beta \wedge (*\alpha) \\ &= - < \beta, \alpha > dvol = - < \alpha, \beta > dvol \end{aligned}$$

If $\pi : P \to M$ is a princ. S^1-bundle, then the $\mathfrak{YM}$-connections on P are the connections whose curvature 2-form is the harmonic rep. of $\mathfrak{c}(P) \in H^2_{dR}(M)$. The curvature form F^w of every w is closed. If $*F^w = +/-F^w$, then $d(*F^w) = 0$ follows from $dF^w = 0$. (bild) If Q_M is negative definite, then $\mathfrak{H}^2_+ = 0$ and $\mathfrak{H}^2 = \mathfrak{H}^2_-$. In this case all $\mathfrak{YM}$- connections on S^1-bundle P are ASD $\mathfrak{YM}-$ connections in the sense that F^w is ASD.
Vorlesung 22:
$\pi : E \to M$ a vector bundle, M oriented 4-dim., Riemannian. $<,>$ a metric on E. Define:

$$* : \Omega^2(E) \to \Omega^2(E) \quad \alpha \wedge *\beta = < \alpha, \beta > dvol$$

where if $\alpha = w \otimes s \in \Omega^2(M) \otimes \Gamma(E)$ and $\beta = \nu \otimes t \in \Omega^2(M) \otimes \Gamma(E)$. Then $\alpha \wedge \beta = < s, t > w \wedge \nu$.

$$** = +1 \text{ on } 2-forms$$

$\Omega^2(E) = \Omega^2_+(E) \oplus \Omega^2_-(E)$ where $\Omega^2_{(+/-)}(E)$ are (+/-1)-eigenvalues of $*$. Suppose $\bigtriangledown$ is a connection on E compatible with $<,>$ in the space that:

$$d < \alpha, \beta > = < \bigtriangledown\alpha, \beta > + < \alpha, \bigtriangledown\beta >$$

$\bigtriangledown$ defines a 1^{st} order D0 from k-forms with values in E to (k+1)-forms with values in E.
If M is compact we can consider L^2-norms on spaces of forms/sections. The formal adjoint of $\bigtriangledown$ w.r.to the L^2-scalar produkts is $+/-*\bigtriangledown *$. Take $P \to M$ to be a princ. G-bundle, $E = P \underset{Ad}{\times} \mathfrak{g}$ with a metric defined by an Ad-invariant scalar product of $\mathfrak{g}$.
If $H = kerw$ is a connection P, then $F^w \in \Omega^2(E)$. w is a $\mathfrak{YM}$-connection if $(D^w)^*F^w = 0$ For M, $(D^w)^*F^w = 0$ follows from $D^w(*F^w) = 0$(YM-eq). If M is 4-dim, then this follows from $*F^w = +/-F^w$(A/SD YM-eq.). The w which solve the (A)SD YM-eq. are called **instantons**. [1^{st} order eq. $\Rightarrow$ 2^{nd} order YM-eq.]

Theorem 9.22. *For any connection w on P, the 4-form $F^w \wedge F^w$ is closed as a 4-form on M. The de Rham coh. class of this form is independent of the choice of w.*

Example 9.23. *1) $G = S^1$: F^w is a closed 2-form on M, its colohomogy class is, up to a constant, the euler class $\mathfrak{e}(P)$. So ,up to a constant, $F^w \wedge F^w$ represents $\mathfrak{e}(P)$.*

*2) $G = SU(n), n \geq 2$: Then cohomology class of $\frac{1}{8\pi^2}F^w \wedge F^w$ is called $c_2(P)$ (the 2^{nd} **Chern class** of P.*

3) $G = U(n), n \geq 2$: The cohomology class of $\frac{1}{8\pi^2}F^w \wedge F^w$ is $c_2(P) - c_1(P)$. [A $U(n)$- bundle over M has Chern classes $c_i(P) \in H^{2i}(M)$]

*4) $G = SO(n), n \geq 3$: The coh. class of $F^w \wedge F^w$ is , up to a constant, $p_1(P)$ (**first Pontryagin class**)*

Proof.

$$d(F^w \wedge F^w) = (\underbrace{D^w F^w}_{=0}) \wedge F^w + F^w \wedge (\underbrace{D^w F^w}_{=0}) = 0 \quad \text{by bianchi}$$

Let w_0 and w_1 be two connection 1-forms on P. Define

$$w_t = tw_1 + (1-t)w_0 \quad \text{for } t \in [0,1]$$

Consider $P \times [0,1] \to M \times [0,1]$ as a princ. G-bundle and w_t as a connection 1-form on this bundle. The curvature $F^{\tilde{w}}$ is a 2-form on $M \times [0,1]$ and $F^{\tilde{w}} \wedge F^{\tilde{w}}$ rep. a de Rham class $k \in H^4(M \times [0,1]$. If i_0, i_1 are the inclusions of M into $M \times [0,1]$ as $M \times \{0\}$ resp. $M \times \{1\}$, then:

$$F^{w_0} \wedge F^{w_0} = i_0^*(F^{\tilde{w}} \wedge F^{\tilde{w}}) \quad F^{w_1} \wedge F^{w_1} = i_1^*(F^{\tilde{w}} \wedge F^{\tilde{w}})$$
$$\Rightarrow [F^{w_0} \wedge F^{w_0}] = i_0^* k = i_1^* k = [F^{w_1} \wedge F^{w_1}]$$

$\square$

$$F^w = \underbrace{\frac{1}{2}(F^w + *F^w)}_{F^w_+ \in \Omega^2_+{}^2(E)} + \underbrace{\frac{1}{2}(F^w - *F^w)}_{F^w_- \in \Omega^2_-(E)}$$

$$\begin{aligned} F^w \wedge F^w &= (F^w_+ + F^w_-) \wedge (F^w_+ + F^w_-) = F^w_+ \wedge F^w_+ + F^w_- \wedge F^w_- \\ &= F^w_+ \wedge *F^w_+ - F^w_- \wedge *F^w_- = |F^w_+|^2 dvol - |F^w_-|^2 dvol \end{aligned}$$

Rewrite:

$$\begin{aligned} \mathfrak{YM}(w) &= \frac{1}{8\pi^2} \int_M |F^w|^2 dvol = \frac{1}{8\pi^2} \int_M |F^w_+|^2 + |F^w_-|^2 dvol \\ &\geq \frac{1}{8\pi^2} | \int_M |F^w_+|^2 - |F^w_-|^2 dvol| = |\frac{1}{8\pi^2} \int_M F^w \wedge F^w| = |c(P)| \\ &\text{e.g.} c = c_2 \text{ if } G = SU(2) \end{aligned}$$

Theorem 9.24. *$\mathfrak{YM}(w) \geq |c(P)|$ with equality $\Leftrightarrow$ w is (A)SD YM.*

If $c(P) \neq 0$, then P has no flat connection. $\mathfrak{YM}$ is minimal precisely for (A)SD YM connections.

Vorlesung 23:

$\pi : P \to M$ a princ. G-bundle. $dim M = 4$, M cpt. oriented, Riemannian. The $SDYM$ equation for w on P is that the $*F^w = F^w \Leftrightarrow F^w_- = 0$. [$ASDYM$ eq. ... $*F^w = -F^w \Leftrightarrow F^w_+ = 0$] Write:

$$F^w = F^w_+ + F^w_-$$

C_P the space of connections on P. $\mathfrak{G}_P$ the gauge group $Aut(P)$.

Definition 9.25.

$$\mathfrak{M} = \{w \in C_P | F^w_+ = 0\} \subset {}^{C_P}/_{\mathfrak{G}_P}$$

the space of gauge equiv. classes of connections on P. The **moduli space** of ASD connections on P.

$$f : C_P \to \Omega^2_+(Ad(P)) \quad w \mapsto F^w_+$$

The ASD connections are precisely the zeroes of f. Take $w \in f^{-1}(0)$. To study $f^{-1}(0)$ locally near w, we need to look at $D_w f$.

Take $A \in T_w C_P = \Omega^1(Ad(P))$:

$$\begin{aligned} D_w f(A) &= \frac{d}{dt} f(w + tA)|_{t=0} = \frac{d}{dt}(F^{w+tA}_+)|_{t=0} \\ &= \frac{d}{dt}(F^w + t \underbrace{D^w A}_{\nabla^w A} + t^2 + ...)_+|_{t=0} \\ &= \frac{d}{dt}(F^w_+ + t(D^w A)_+ + t^2 ...)|_{t=0} = (D^w A)_+ \end{aligned}$$

$$D_w f : \Omega^1(Ad(P)) \to \Omega^2_+(Ad(P)) \quad A \mapsto (D^w A)_+$$

This may or may not be surjektive. In any case, its kernel is infinit-dimensional because it contains $T_w G(w)$.

$$[w] = G(w) \in {}^{C_P}/_{\mathfrak{G}_P} \quad T_{[w]}{}^{C_P}/_{\mathfrak{G}_P} = {}^{T_w C_P}/_{T_w \mathfrak{G}_P(w)}$$

$T_e \mathfrak{G}_P = \Omega^0(Ad(P)) \ni s$, $x \in M$:

$$exp(ts(x)) \text{ acts on } P \text{ by } exp(ts(x))(p) = p \cdot exp(ts(x))$$

$$o : \mathfrak{G}_P \to C_P \quad \psi \mapsto \psi(w)$$

parametrises the $\mathfrak{G}_P$-Orbit of w:

$$T_w \mathfrak{G}_P(w) = Im(D_e o)$$

$U_t = exp(ts(x)), o(U_t) = U_t w$:

$$\begin{aligned}(D_e o)(s) = D_e o(\dot{u}(0)) &= \frac{d}{dt} o(u_t)|_{t=0} = \frac{d}{dt}(u_t(w))|_{t=0} \\ &= \frac{d}{dt}(u_t^{-1} \bigtriangledown^w u_t)|_{t=0}\end{aligned}$$

[Remember: $\psi : P \to P, \psi^* w = Ad(u^{-1})w + u^*\theta, \psi(p) = p \cdot u(p)$]. If $\bigtriangledown^w$ is covariant derivative induced by w, then $\bigtriangledown^{u(w)}$ is given by:

$$\begin{aligned}\bigtriangledown^{u_t(w)}(s') &= u_t^{-1} \bigtriangledown^w u_t(s') = u_t^{-1}(u_t \bigtriangledown^w s' + (\bigtriangledown^w u_t)s') \\ &= \bigtriangledown^w s' + u_t^{-1}(\bigtriangledown^w_{u_t})s' = \bigtriangledown^w s' + (u_t^{-1} du_t)s' \\ \Rightarrow \bigtriangledown^{u_t(w)} &= \bigtriangledown^w + u_t^{-1} du_t\end{aligned}$$

$u_t^{-1} u_t = e \Rightarrow u_t^{-1} du_t + d(u_t^{-1})u_t = 0$ differentiating this: $(\dot{u_t^{-1}}) \cdot u_t + u_t^{-1} \cdot \dot{u}_t = 0 \Rightarrow (\dot{u_t^{-1}}) = -u_t^{-1} \dot{u}_t u_t^{-1}$ [

$$\begin{aligned}\frac{d}{dt} \bigtriangledown^{u_t(w)} (s')|_{t=0} &= \dot{u}_t^{-1}(\bigtriangledown^w u(t))(s') + u_t^{-1}(\bigtriangledown^w \dot{u}_t)(s')|_{t=0} \\ &= -s \bigtriangledown^w u \bigtriangledown^w s\end{aligned}$$

]

$$\begin{aligned}\frac{d}{dt} \bigtriangledown^{u_t(w)} |_{t=0} &= (\frac{d}{dt}(u_t^{-1}) \underbrace{du_t} = 0 + u_t^{-1} \frac{d}{dt}(du_t))|_{t=0} \\ &= 0 + \bigtriangledown^w s\end{aligned}$$

$$\Omega^0(Ad(P)) \underset{D_e o}{\to} \Omega^1(Ad(P)) \underset{D_w f}{\to} \Omega^2_+(Ad(P)) \quad \mapsto A \mapsto (D^w A)_+$$

We have $Im(D_e o) \subset ker(D_w f)$:

$$H_w^0 := ker(D_e o) = ker(\bigtriangledown^w) \quad H_{w:}^1 = {}^{ker(D_w f)}/_{Im(D_e o)} \quad H_w^2 := coker D_w f$$

<u>Fact:</u> The H_w^i are finite-dimensional, because the $ASDYM$ modulo gauge is an elliptic PDE.

$$dim H_w^1 - dim H_w^0 - dim H_w^2 = dimker(D_w f \oplus (D_e o)^*) - dimcoker(D_e f \oplus (D_e o)^*)$$

$$= \text{fredholm index of } D_w f \oplus (D_e o)^*$$

= expression in top. invariants of M and P given by the "Atiyah-singer index theorem"

$$\Omega^1(Ad(P)) \underset{(D_e o)^* \oplus D_w f}{\rightarrow} \Omega^0(Ad(P)) \oplus \Omega^2_+(Ad(P))$$

(H^1, H^2 vanish: nice situation)
<u>*Vorlesung 24:*</u>

$$\Omega^0(Ad(P)) \rightarrow \Omega^1(Ad(P)) \underset{D_w f}{\rightarrow} \Omega^2_+(Ad(P))$$

M compact oriented Riemannian mfd., $dim M = n$

$$0 \rightarrow \Omega^0(M) \underset{d}{\rightarrow} \Omega^1(M) \underset{d}{\rightarrow} ... \rightarrow \Omega^{n-1}(M) \underset{d}{\rightarrow} \Omega^n(M) \rightarrow 0$$

$d^2 = 0 \Rightarrow$ this is a complex, called the **de Rham complex.of M. Hodge decomposition** for the de Rham complex:

$$\Omega^k(M) = \underbrace{\mathfrak{H}^k \oplus Imd}_{kerd} \oplus Imd^*$$

$\Rightarrow$ every de Rham coh.class has a unique harmonic rep.

$$\mathbb{X}(M) = \sum_{k=0}^{n} (-1)^k dim H^k_{dR}(M)$$

$$\Omega^{even}(M) = \bigoplus_k \Omega^{2k}(M) \quad \Omega^{odd}(M) = \bigoplus_k \Omega^{2k+1}(M)$$

$$d \oplus d^* : \Omega^{even}(M) \rightarrow \Omega^{odd}(M) \quad ker(d \oplus d^*) = \bigoplus_l \mathfrak{H}^{2k} \quad coker(d \oplus d^*) = \bigoplus_k \mathfrak{H}^{2k+1}$$

$$ind(d \oplus d^*) = dimker(d \oplus d^*) - dimcoker(d \oplus d^*) = \sum_{k=0}^{n} (-1)^k \mathfrak{H}^k = \mathbb{X}(M)$$

Assume n=4:

$$\Omega^0(M) \underset{d}{\rightarrow} \Omega^1(M) \underset{d_+}{\rightarrow} \Omega^2_+(M) \quad d_+ = \pi_+ \circ d$$

$$\pi_+ : \Omega^2(M) \rightarrow \Omega^2_+(M) \quad ker\pi = \Omega^2_-(M)$$

Proposition 9.26. *This is a complex, whose cohomology is:* $H^0_{dR}(M), H^1_{dR}(M), \mathfrak{H}^2_+$. *This cx. is called the* ***half de Rham complex****.*

Proof. $d_+ \circ d = \pi_+ \circ \underbrace{d \circ d}_{=0} = 0$ so we do have a complex.

$H^0 = ker(d : \Omega^0(M) \to \Omega^1(M)) =$ the vector space of constant funtions on M

$$= \mathbb{R} \text{ if } M \text{ is connected}$$

if $\alpha \in \Omega^1(M)$, then:

$$0 \underset{stokes}{=} \int_M d(\alpha \wedge d\alpha) = \int_M d\alpha \wedge d\alpha = \int_M (d_+\alpha + d_-\alpha) \wedge (d_+\alpha + d_-\alpha)$$
$$= \int_M \underbrace{|d_+\alpha|^2}_{=0} dvol - \int_M |d_-\alpha|^2 dvol$$

So $d_+\alpha = 0$ implies $d_-\alpha = 0$ and therefore $d\alpha = 0$. $\Rightarrow$ The coh. of the half de Rham cx. in the middle is $H^1_{dR}(M)$.
Let $h \in \mathfrak{H}^2_+$.

$$< h, d_+\alpha >_{L^2} = \int_M < h, d_+\alpha > dvol = \int_M h \wedge (d_+\alpha + d_-\alpha) = \int_M h \wedge d\alpha$$
$$= \int_M d(h \wedge \alpha) \underset{stokes}{=} 0$$
$$\Rightarrow Imd_+ \text{ is } L^2 - \text{orthogonal to } \mathfrak{H}^2_+$$

Take $w \in \Omega^2_+(M) \subset \Omega^2(M)$. w has a Hodge decomposition: $w = h + d\alpha + d^*\beta$. Apply $*$ to get:

$$w = *w = *h + \underbrace{*d^*\beta}_{\in Imd} + \underbrace{*d\alpha}_{\in Imd^*}$$

By uniqueness of the Hodge decomposition we have $*h = h$, so $h \in \mathfrak{H}^2_+$, $*d^*\beta = d\alpha$ and $*d\alpha = d^*\beta$

$$\Rightarrow w = h + d\alpha + *d\alpha = h + \frac{1}{2}(Id + *)d\alpha) = h + d_+\alpha$$

$\Rightarrow coker d_+ = \mathfrak{H}^2_+$ ☐

In the case of the half de Rham cx we have computed:

$$dimH^1 - dimH^0 - dimH^2 = dimH^1_{dR}(M9 - dimH^0_{dR}{}^0(M) - dim\mathfrak{H}^2_+ = b_1(M) - b_0(M) - b_2^+(M)$$

$$H^2_{dR}(M) = \mathfrak{H}^2 = \mathfrak{H}^2_+ \oplus \mathfrak{H}^2_- \quad b_2(M) = b_2^+(M) + b_-^2(M)$$

Definition 9.27.

$$\sigma(M) = b_2^+(m) + b_2^-(M)$$

the **signature of** M

$$\begin{aligned} b_1(M) - b_0(M) - b_2^+(M) &= \frac{1}{2}(2b_1(M) - 2b_0(M) - 2b_2^+(M)) \\ &\underset{P.D.}{=} \frac{1}{2}(b_1(M) + b_3(M) - b_0(M) - b_4(M) - (b_2(M) + \sigma(M))) \\ &= -\frac{1}{2}(\mathbb{X}(M) + \sigma(M)) \end{aligned}$$

$$\Omega^2(Ad(P)) \to \Omega^1(Ad(P)) \underset{D_wf}{\to} \Omega^2_+(Ad(P))$$

If $Ad(P)$ is trivial, then $dimH_0^1 - dimH_w^0 - dimH_w^2 = -\frac{1}{2}(\mathbb{X}(M) + \sigma(M)) \cdot dimG$
For general P one gets from the Atiyah-singer index theorem:

$$v - dim\mathfrak{M} = dimH_w^1 - dimH_w^0 - dim_w^2 = 8k - \frac{1}{2}dimG(\mathbb{X}(M) + \sigma(M))$$

where $k \in \mathbb{Z}$ is a top invariant of P.

Example 9.28. *$G = SU(2) \Rightarrow dimG = 3$: In this case $k = -c_2(P)$, with $c_2(P) = \frac{1}{8\pi^2}\int_M F^w \wedge F^w$ for any connection w on P. If P admits an ASD connection, then $c_2(P) = \frac{1}{8\pi^2}\int_M F^w_- \wedge F^w_- = -\frac{1}{8\pi^2}\int_M |F^w_-|^2 dvol \leq 0$
For $G = Su(2), k = +1$ we get:*

$$\begin{aligned} v - dim\mathfrak{M} &= 8 - \frac{3}{2}(\mathbb{X}(M) + \sigma(M)) \\ &= 8 - 3(1 - b_1(M) + b_2^+(M)) \end{aligned}$$

using: M connected $\Rightarrow b_0 = 1$. In particular, if $b_1(M) = 0$ and $b_2^+(M) = 0$ then $v - dim\mathfrak{M} = 5$

[dim 4: YM equationio reducedto SD, ASD (no relatable in other)]

$$\nu : \mathfrak{G}_P \to C_P \quad \psi \mapsto \psi(W)$$

G acts on M the G-orbit of p is $G(p)$, diffeomorphic to ${}^G/_H$ where $H = stab(p)$.

Definition 9.29.

$$Stab(w) = \{\psi \in \mathfrak{G}_P | \psi(w) = w\} \quad \textbf{(Stabiliser group)}$$

Lemma 9.30. *Assume M is connected. Write $\psi(p) = p \cdot u(p)$. Then $\psi \in stab(w) \Leftrightarrow u$ is a connected map into the centralizer of $Hol(H, p) \subset G$, $H = kerw$.*

Proof. If $\psi(w) = w$, then ψ preserves the holonomy subbundles of $H = kerw$.

$$\psi^*(w) = Ad(u^{-1})w + u^*\theta = w$$

gives us:

- u constant
- $Ad(u^{-1})w = w$

$\Rightarrow$ the value of u comutes with elements of the holonomy group. $\square$

Example 9.31. *The constant maps u to $C(G) \subset G$, the center of G, are always in $stab(w)$ for all w. In particular, if G is Abelian, then $C(G) = G$, so $stab(w) = G$ for all w. If $G = SU(2)$, then $C(SU(2)) = \mathbb{Z}_2$.*

Vorlesung 25:
$\pi : P \to M^4, w$ satisfyies $*F^w = -F^w$ so w is ASDYM.

$$\Omega^0(Ad(P)) \underset{D_0}{\to} \Omega^1(Ad(P)) \underset{D_1}{\to} \Omega^2_+(Ad(P))$$

$$H^0_w \quad H^1_w \quad H^2_w$$

$$\Omega^0(Ad(P)) = Lie(\mathfrak{G}) \supset kerD_0 = L(stab(w))$$

The expected dimesnion of $\mathfrak{M}$ is $dimH^1_w - dimH^0_w - dimH^2_w$. In sufficiently nice situations $H^0_w = 0 = H^2_w$. By the vanishing of H^2_w, w is a transverse or regular zero of the map:

$$C \to \Omega^2_+(Ad(P)) \quad w \mapsto F^w_+$$

Then $\mathfrak{M}$ is a mfd. of dimension equal to $dimH^1_w$.

$$f : P \to Q \quad f(p) = q \quad D_pf \text{ not surjective}$$

If $dimP \geq dimQ$, we cab perturb f sligtly, so that $\forall p \in f^{-1}(q) : D_pf$ is surjective.

In our situation for ASDYM eq. one can often achieve surjectivity of D_1 by perturbing the Riemannian metric on M.

Definition 9.32. w is called reducible if $stab(w)$ has positive dimension, equivalently $H^0_w \neq 0$. If $H^0_w = 0$, but $H^0_w \neq 0$, then locally near w, $\mathfrak{M}$ looks like the zero-set of a map $\psi : H^1_w \to H^2_w$ with $D_0\psi = 0$.

If both H^0_w and H^2_w are $\neq 0$, then $stab(w)$ acts on H^1_0 on H^2_w (normal case H^i_0 are products of $\mathbb{C}$ and $stab(w)$ is S^1 acting on them by rotation) and $\psi : H^1_w \to H^2_w$ is $stab(w)$-equivariant, and a neighbpurhood of $[w]$ in $\mathfrak{M}$ looks like ${\psi^{-1}(0)}/{stab(w)}$.

Example 9.33. *$G = S^1 : G = C(G)$ the Ad-representation is trivial so $Ad(P)$ is the trivial bundle.*

$$\Omega^0(M) \to \Omega^1(M) \to \Omega^2_+(M)$$

For any w this is the deformation cx. with cohomology :

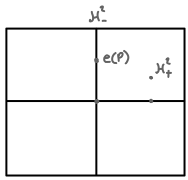
$$H^0_w = \mathbb{R} \quad H^1_{dR}(M) \quad \mathfrak{H}^2_+$$

The expected dimension of $\mathfrak{M}$ is $b_1(M) - 1 - b_2^+(M)$ For S^1 the ASDYM connections are exactly, those w for which F^w is an ASD 2-form representing $\mathfrak{e}(P)$. The space of gauge equivalence classes such connections $\mathfrak{M}$ is either empty, or a torus of dimesnnion $b_1(M)$.

case 1: $b_2^+ = 0$ *In this case $\mathfrak{e}(P)$ is represented by a unique ASD harmonic form $\Rightarrow \mathfrak{M} \neq \emptyset$. Therefore the moduli space is a torus of dimension $b_1(M)$. The 'expected' dimension is $b_1(M) - 1$. Here $stab(w) = S^1$ and this acts trivially H^1_w*

case 2: $b_2^- = 0$, *$H^2_{dR}(M) = \mathfrak{H}^2 = \mathfrak{H}^2_+$. If $\mathfrak{e}(P) \neq 0$, then $\mathfrak{e}(P)$ cannot ve represented by an ASD harmonic form, so there are no solution to the ASDYM equation. If $\mathfrak{e}(P) = 0$, then the ASDYM connections are exactly the flat connections.*

Case 3: $b_2^+ > 0, b_2^- > 0$ *$H^2_{dR}(M) = \mathfrak{H}^2 = \mathfrak{H}^2_+ \oplus \mathfrak{H}^2_-$. $\mathfrak{M} = \emptyset \Leftrightarrow \mathfrak{e}(P) \in \mathfrak{H}^2_-$. In this*



case, given P, one can pertube the metric so that $\mathfrak{e}(P) \notin \mathfrak{H}^2_-$

Example 9.34. *$G = SU(2), C(G) = \mathbb{Z}_2$: One often replaces $\mathfrak{G}$ by $\mathfrak{G}/\mathbb{Z}_2$. If a connection w on an $SU(2)$-bundle P reduces to S^1, then we can discuss wether it is ASDYM trough the aboce dimension for S^1.*
we can look at a reduction :

$$\begin{array}{ccc} MQ & \longrightarrow & P \\ & {\scriptstyle S^1} \searrow \quad \swarrow {\scriptstyle SU(2)} & \\ & M & \end{array}$$

in terms of the associated vector bundles.

$$E = P \underset{\rho}{\times} \mathbb{C}^2$$

where $\rho : SU(2) \to GL_2(\mathbb{C})$ *is the standard rep. of* $SU(2)$*.*

$$U(1) = S^1 \to SU(2) \quad a \mapsto \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix}$$

P$reducesto$S^1 if and only if $E = L \oplus L^{-1}$.
If $L \to M$ is a rank 1 vector bundle, a cx. line bundle, then $L_{\mathbb{R}}$ is a oriented rank 2 real vector bundle. The orientation is defined by saying that multiplication by $i \in \mathbb{C}$ in a fibre of L is rotation by +90 in $L_{\mathbb{R}}$. Conversely, if V is an oriented rank 2 vector bundle. We define multiplication by $i \in \mathbb{C}$ to be the rotation by +90. Then we extend linearly to $a + ib$. This makes V into a cx. line bundle L with $L_{\mathbb{R}} \cong V$. This equivalence comes from $SO(2) = U(1)$. The cx. line bundles over M form a group w.r.t. $\otimes$. The neutral element is the trivial bundle and the inverse L^{-1} of L is $Hom(L, \mathbb{C}) = L^*$.

Definition 9.35. $c_1(L) := \mathfrak{e}(L_{\mathbb{R}}) \in H^2(M)$ the **first chern class** of L.

$$Vect_1(M, \mathbb{C}), \otimes) \underset{c_1, \cong}{\to} (H^2(M, \mathbb{Z}), +)$$

$[Vect_1(M, \mathbb{C}), \otimes) =$ iso. classes of cx. rank 1 vector bundle on $M]$ This is a homomorphism and in fact an isomorphism.
In general, for cx. vector bundles of arbitrary rank, we have $c_1, c_2, ...$

$$\mathfrak{c}(E) = 1 + c_1(P) + c_2(E) + ... \in \bigoplus_{i=0}^{\infty} H^{2i}(M; \mathbb{Z})$$

if $E = E_1 \oplus E_2$ then $\mathfrak{c}(E) = c(E_1) \cup c(E_2)$.
A cx. vector bundle E has structure group $SU(k)$ instead of $U(k)$, $k = rkE$ $\Leftrightarrow c_1(E) = 0$.
Assume $rk(e) = k = 2$. Then E is an $SU(2)$-bundle $\Leftrightarrow c_1(E) = 0$. Then, in this case, assume $E = L_1 \oplus L_2$. Then:

$$0 = c_1(E) = c_1(L_1) + c_1(L_2) \Rightarrow c_1(L_2) = -c_1(L_1) \Rightarrow L_2 = L_1^{-1}$$

$$c_2(E) = c_2(L_1) \cdot 1 + c_1(L_1) \cdot c_1(L_2) + 1 \cdot \underbrace{c_2(L_2)}_{=0 \text{ since } 2 > rkL_2} = -c_1^2(L) \text{ where } L = L_1$$

Hence $c_2(E) \in H^4(M, \mathbb{Z}) = \mathbb{Z}$ if M is connected, cpt., oriented 4-mfd.

$$\frac{1}{8\pi^2} \int_M F^w \wedge F^w = k \quad \text{if } c_2(E) = k \in \mathbb{Z}$$

Fact: $SU(2)$-bundles over M^4 are classified up to isomorphism by $c_2(E)$. In particular, for every k there is a $SU(2)-$bundle, unique up to isomorphism. The 'expected' dimension of $\mathfrak{M}$ for an $SU(2)$-bundle is $8\underbrace{k}_{=1} - 3(1 \underbrace{-b_1}_{=0} + \underbrace{b_2^+}_{=0}) =$ 5

Vorlesung 26:

$\pi : P \to M, M^4$ a cpt. oriented smooth 4-mfd. the princ. $SU(2)$-bundle with $c_2(P) = -1$. Assume: $b_2^+(M) = 0$ and $\pi_1(M) = \{1\}$, so M is simply connected. If we pick a Riemannian metric on M we can study $\mathfrak{M} := {}^{\{w \in C_P \mid *F^w = -F^w\}}/_{\mathfrak{G}}$. The expected dimension of $\mathfrak{M}$ is

$$8k - 2(1 - b_1(M) + b_2^+(M)) = 8 \cdot 1 - 3(1 - 0 + 0) = 5$$

$E_P \underset{\rho}{\times} \mathbb{C}^2$. For every splitting $E = L \oplus L^{-1}$ there is a reducible ASD connection. Since $b_1(M) = 0$, this is unique up to gauge equivalence. We have

$$+1 = c_2(E) = -c_1(L)^2 = -Q_M(c_1(L), c_1(L))$$

where $G_M : H^2(M;\mathbb{Z}) \times H^2(M;\mathbb{Z}) \to \mathbb{Z}, (\alpha, \beta) \mapsto < \alpha \cup \beta, [M] >$

$$[c_2(P) = \frac{-1}{8\pi^2} \int_M F^w \wedge F^w = \frac{-1}{8\pi^2} \int_M |F_+^w|^2 - |F_-^w|^2 dvol \geq 0 \quad \text{if } P \text{ admits an ASD con.}]$$

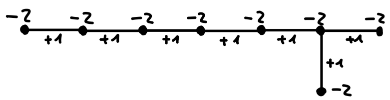
For every $\alpha \in H^2(M, \mathbb{Z})$ there is a unique cx. line bundle $L \to M$ with $c_1(L) = \alpha$. The numver of reductions of E to $L \oplus L^{-1}$ is given by the numver of pairs $+/-\alpha \in H^2(M, \mathbb{Z})$ with $\alpha^2 = Q_M(\alpha, \alpha) = -1$. The number of pairs is $\leq rkQ_M = b_2(M) = b_2^-(M)$

Example 9.36. $M = S^2 \times S^2$: $Q_{S^2 \times S^2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $w.r.t.$x$_1, x_2$ given by the two factors.

$$Q_{S^2 \times S^2}(\alpha, \alpha) = Q_{S^2 \times S^2}(ax_1 + bx_2, ax_1 + bx_2) = 2ab \quad a, b \in \mathbb{Z}$$

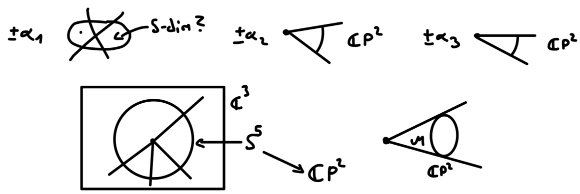
This $Q_{S^2 \times S^2}$ is evenm so there are no $\alpha \in H^2(S^2 \times S^2)$ with $\alpha^2 = +/-1$

Example 9.37. E_8



$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

this is negative definite, so if $Q_M = E_8$, then $b_2^+(M) = 0$. E_8 is even, so there would be no α with $\alpha^2 = +/-1$. If we linearize at a reducible ASD connection we get:



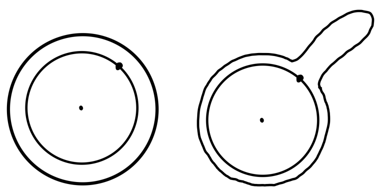
$$\Omega^0(Ad(P)) \to \Omega^1(Ad(P)) \to \Omega^2_+(Ad(P))$$

$$\mathbb{R} = H^0_w \qquad h^1_w = \mathbb{R}^6 = \mathbb{C}^3 \qquad H^2_w = 0$$

A neighbourhood of $[w]$ in $\mathfrak{M}$ looks like ψ^{-1}/S^1 for an S^1-equivariant map $\psi : H^1_w \to H^2_w.H^2_w = 0$ implies $\psi \equiv 0$, so

$$\psi^{-1}/S^1 = \mathbb{C}^3/S^1 = cone(\mathbb{C}P^2)$$

Example 9.38. *$M = S^4, g = g_0$(standard metric)$S^4 = SO(5)/SO(4)$: $H^2(S^4) = 0 \Rightarrow$ there are no reducible solutions. However, there are irreducible ones! There is a homogeneous solution (looks same at every point):*

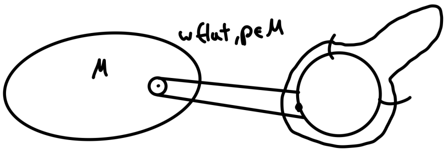


$$\mathfrak{M} = Conf(S^4, g_0)/Isom(S^4, g_0) = SO(5,1)/SO(5) \cong B^5$$

In some sence $\partial\bar{\mathfrak{M}} = S^4$, where $\bar{\mathfrak{M}}$ is a compactification of $\mathfrak{M}$.

Theorem 9.39 (Uhlenbeck weak compactness). *The only non-compactness in $\mathfrak{M}$ arises from concentration of curvature at isolated points in M.*

For the 1-instanton moduli space there can be only one point of concentration.



Theorem 9.40 (Taubes existence theorem). *If M^4 is closed, oriented with a Riemannian metric and $b_2^+(M) = 0$, then there exists irrreducible ASD connections on the 1-instanton bundle obtained by gluing concentrated 1-instantons on S^4 into the flat connection on M.*

One splices together the flat connection on M with an instanton on S^4 to obtain an approximate solution to the ASDYM eq. on $M^4 \# \mathrm{S}^4$.Becouse $b_2^+(M) = 0$, this approximate solution can be perturbed to an actual solution.