Foliations and Foliated Vector Bundles

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Abstract

The following is a revised version of lectures given at MIT during the Fall Term of 1969. Its eventual goal is to describe the classifying space for codimension q foliated manifolds which has recently been constructed by A. Haefliger.

Contents

§1. Definitions, Examples, References

Intuitively, a manifold M of dimension m is “foliated” if it is expressed as the union of l-dimensional submanifolds which fit alongside each other, locally, like parallel l-planes in Euclidean m-space. No restriction is to be placed on the global behavior of these submanifolds. (Compare 1.4 and 1.6). For that reason, some care will be needed in formulating a precise definition.

We begin by considering the disjoint union of all of the submanifolds in question. This union L can be topologized naturally as an l-dimensional manifold. Note that the natural mapping $L \to M$ is bijective (i.e., it is one-to-one and onto).

We will assume that $l < m$, since the case $l = m$ is uninteresting. It follows that the manifold L must be an enormous object: it can never have a countable basis, and will usually have uncountably many components.

Definition 1.1. A **foliation** of a topological manifold M consists of a manifold L of smaller dimension, together with a continuous bijective mapping

$$f : L \to M$$

which satisfies the following flatness condition. Each point of M should possess an open neighborhood U so that:

(1) U is homeomorphic to a convex open subset of the Euclidean space $\mathbb{R}^m$, under a homeomorphism h;

(2) the mapping f carries each component of $f^{-1}(U)$ homeomorphically to a submanifold of U; and

(3) this family of submanifolds corresponds, under h, precisely to a family of parallel l-planes in Euclidean space, intersected with the open set $h(U)$.

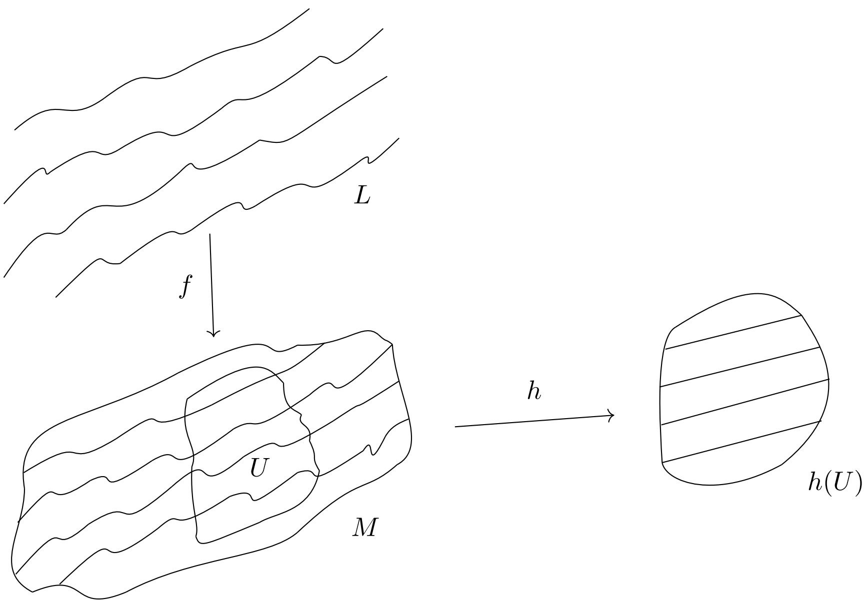

Figure 1

Two such foliations $f : L \to M$ and $f' : L' \to M$ are **isomorphic** if there exists a homeomorphism $L \leftrightarrow L'$ so that the triangle

$$\begin{array}{ccc} L & \longleftrightarrow & L' \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

is commutative. Of course we cannot distinguish in any way between two isomorphic foliations of M.

Each component of L is called a **leaf** of the foliation, and L itself is called the **leaf manifold**. (It is often convenient to identify a leaf with its image in M; Although this practice is dangerous.) The dimension l of L is called the **dimension** of the foliation, and the difference $m - l$ is called the **codimension**.

Remark. In order to specify a foliation on a manifold M, it suffices to specify the foliation on a small neighborhood of each point of M. In fact, if we are given foliations of open subsets U_α covering M, and if these foliations coincide up to isomorphism on the intersections $U_\alpha \cap U_\beta$, then there is one and, up to isomorphism, only one foliation of M which restricts to the given foliations of the U_α. The proof will be left the reader.

So far we have spoken only of topological foliations. Given a foliation $f : L \to M$ where M is a smooth manifold of class $\mathcal{C}^r$, we can require that the coordinate charts h of L should be $\mathcal{C}^r$-diffeomorphisms. If such $\mathcal{C}^r$-coordinate charts, compatible with the foliation, exist about every point of M, then f is called a **smooth foliation** of class $\mathcal{C}^r$. It then follows that the leaf manifold L possesses a unique $\mathcal{C}^r$-smoothness structure so that f is a $\mathcal{C}^r$-immersion. Similarly one can define the concept of a piecewise-linear foliation, or a real analytic[1] foliation, or a complex analytic foliation.

In practice, we will concentrate on the theory of smooth foliations, especially those of class $\mathcal{C}^\infty$.

Example 1.2. Any manifold M can be foliated into points. That is, we let L be the unique 0-dimensional manifold which maps bijectively to M. In spite of its unprepossessing appearance, this pointwise foliation will play a significant role in subsequent sections.

Example 1.3. The plane $\mathbb{R}^2$ can be foliated into the curves

$$y = \log|\sec x| + c,$$

together with the vertical lines $\cos x = 0$. (See Figure 2).

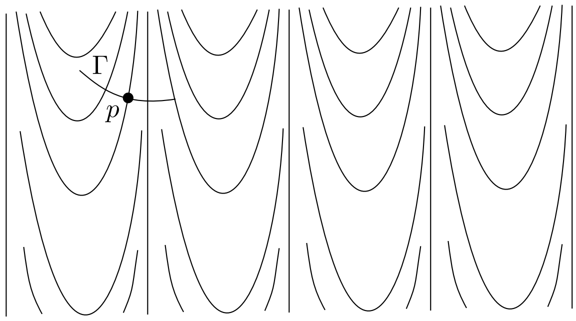

Figure 2

This corresponds to the set of solution curves to the differential equation.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tan x.$$

In this example, each leaf is globally very well behaved. In fact, one can construct a small transverse curve Γ, through any point p, which intersects each leaf at most once. With this fact in mind, one is tempted to look at the quotient space (perhaps it should be called M/L) in which two points of plane are identified if and only if they lie on a common leaf. (Compare Palais [11]). This quotient space does indeed exist, and is locally Euclidean in our example. But it fails to satisfy the Hausdorff axiom.

Example 1.4. Let $\mathbb{R}^2/\mathbb{Z}^2$ denote the torus which is obtained from the plane by identifying two points if and only if they are congruent modulo the lattice $\mathbb{Z}^2$ of integer points. For each fixed slope s, this torus can be foliated into lines $y = sx + c$ (corresponding to the differential equation $\mathrm{d}y/\mathrm{d}x = s$). If s is rational, then each leaf is compact. But if s is irrational, each leaf is everywhere dense in the torus.

Example 1.5. According to Reeb, the sphere S^3 possesses a 2-dimensional foliation which is smooth of class $\mathcal{C}^\infty$. This foliation can be described roughly as follows. The interior of a solid torus is foliated into 2-cells, each of which winds asymptotically to the boundary torus, as sketched in Figure 3. (Each leaf can be thought of as a snake trying to swallow its own tail. Our snakes, being $\mathcal{C}^\infty$, do not have pointed tails.) The complementary solid torus is foliated in an analogous manner. Thus the common boundary torus is the only compact leaf.

[1]Compare Example 1.5.

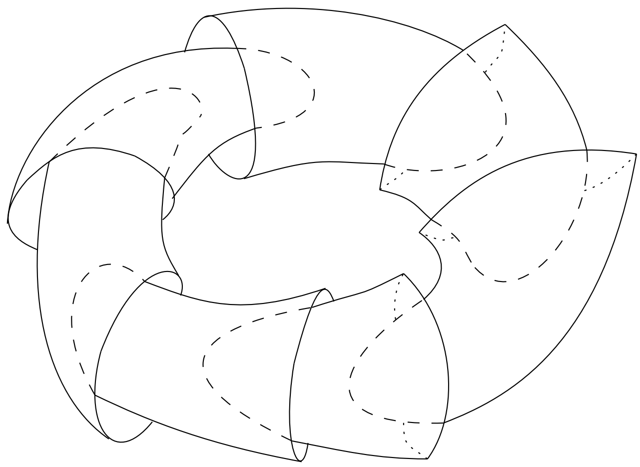

Figure 3

By way of contrast, Haefliger has proved that S^3 does not possess any 2-dimensional real analytic foliation. **Thus there is an essential difference between the theory of $\mathcal{C}^\infty$-foliations and the theory of real analytic foliations.** I do not know whether or not there is any analogous essential difference between $\mathcal{C}^1$-foliations and $\mathcal{C}^\infty$-foliations.

Example 1.6. To conclude, we describe a pathological example, consisting of a 2-dimensional foliation of a 3-manifold with only one leaf. The main object of this exercise is to imbue the reader with suitable respect for non-paracompact manifolds.

Let the index α range over the real numbers. Our manifold M is to be covered with coordinate patches $g_\alpha(\mathbb{R}^3)$, each diffeomorphic with Euclidean 3-space. The image $g_\alpha(x_\alpha, y_\alpha, z_\alpha)$ is to be identified with $g_\beta(x_\beta, y_\beta, z_\beta)$, where $\alpha \neq \beta$, if and only if

$$\begin{aligned} &x_\beta = x_\alpha \neq 0 \text{ (call this common value } x), \\ &y_\beta = y_\alpha + (\beta - \alpha)/x, \text{ and} \\ &z_\beta = x_\alpha + (\beta - \alpha) \operatorname{sgn} x. \end{aligned}$$

Evidently these formulas define a diffeomorphism

$$h_{\beta\alpha} : (x, y_\alpha, z_\alpha) \mapsto (x, y_\beta, z_\beta)$$

from the open set $x \neq 0$ to itself. Since the consistency conditions

$$\begin{aligned} &h_{\gamma\beta} \circ h_{\beta\alpha} = h_{\gamma\alpha} \text{ for } \gamma \neq \alpha, \\ &h_{\alpha\beta} \circ h_{\beta\alpha} = \text{identity}, \end{aligned}$$

are satisfied, it follows that the quotient space M is well defined, and locally Euclidean.

In fact, M also satisfies the Hausdorff axiom. The verification of this fact will be left to the reader.

Finally, note that the foliation of $g_\alpha(\mathbb{R}^3)$ into leaves z_α = constant is preserved by the coordinate transformations $h_{\beta\alpha}$. Hence these foliations piece together to yield a foliation of M.

Given $\alpha \neq \beta$, note that each leaf $z_\alpha = c$ in $g_\alpha(\mathbb{R}^3)$ intersects both the leaf $z_\beta = c + \alpha - \beta$ and the leaf $z_\beta = c + \beta - \alpha$ in $g_\beta(\mathbb{R}^3)$. Fixing β, and choosing α and c at will, it follows easily that the leaf manifold L is connected.

Remark. This pathology can never occur if the manifold M is paracompact. For then M can be given a Riemannian metric, hence L inherits a Riemannian metric, hence every leaf L_0 has a countable basis. It follows that no leaf can intersect a coordinate patch in M more than a countable number of times. (Compare the page 94-98 of Chevalley [9]).

References

An excellent survey of the current state of the art is given in §4 of the following.

[1] E. Thomas. “Vector Fields on Manifolds”. In: *Bulletin of the American Mathematical Society* 75 (1969), pp. 643–683.

A complete bibliography is included, the following expositions are also noteworthy:

[2] G. Reeb. *Sur Certaines Propriétés Topologiques des Variétés Feuilletées*. Paris: Hermann, 1952.

[3] A. Haefliger. “Variétés Feuilletées”. In: *Annali della Scuola Normale Superiore di Pisa*. 3rd ser. 16 (1962). Also published in *Topologia Differenzibile*, Centro Internazionale Matematico Estivo, Urbino, 1962, Edizioni Cremonese, Rome, pp. 367–397.

[4] S. P. Novikov. “The Topology of Foliations”. In: *Transactions of the Moscow Mathematical Society* (1965). AMS translation, pp. 145–168.

Also, for concise descriptions of the literature, one should not overlook:

[5] N. Steenrod. “Reviews of Papers in Algebraic and Differential Topology I”. In: *Proceedings of the International Congress of Mathematicians*. American Mathematical Society, 1968, pp. 848–856.

Further references will be given at the end of each section.

§2. A Historical Note

Any smooth foliation can be described as the set of solutions to an associated system of differential equations on M. (Compare Example 1.3 and 1.4). If the foliation dimension is 2 or more, then these differential equations are "overdetermined", so that appropriate integrability conditions must be satisfied.

These facts are described in a classical result which is often called the "Frobenius Theorem". (Compare Frobenius [8], published in 1877). Actually, as Frobenius himself pointed out, the theorem in question had been proved a decade earlier by A. Clebsch. In fact, a recognizable version had been proved already in 1840, by F. Deahna.

It is sad to relate that Deahna did not profit by being so far ahead of his time. According to the entry in Poggendorff, Deahna had barely attained the rank of "Hülfslehrer" in a secondary school when he died in 1844, at the age of 28.

Here is a modern formulation of the Deahna-Clebsch-Frobenius theorem. Any smooth foliation

$$f : L \to M,$$

of class $\mathcal{C}^1$, determines a linear mapping $\mathrm{d}f_x$ from each tangent vector space TL_x to the tangent vector space $TM_{f(x)}$. Setting $f(x) = y$, the image $\mathrm{d}f_x(TL_x)$ will be denoted by

$$\Phi_y \subset TM_y.$$

Thus the correspondence

$$y \mapsto \Phi_y$$

describes a continuous field of tangent l-planes on M.

Evidently, the foliation is uniquely determined by this tangent l-plane field. If the foliation is smooth of class $\mathcal{C}^{r+1}$, note that the l-plane field is smooth of class $\mathcal{C}^r$. (That is, it is spanned locally by l independent vector fields of class $\mathcal{C}^r$).

We will say that a vector field v on M is a **section** of the l-plane field Φ if $v(y) \in \Phi_y$ for all y.

Theorem. A smooth l-plane field Φ of class $\mathcal{C}^r$, $r \geqslant l$, is tangent to a (necessarily unique) foliation of M if and only if the following integrability condition is satisfied. For any $\mathcal{C}^1$-vector fields v and w on M which are sections of Φ, the bracket $[v, w]$ should also be a section of Φ. This foliation, when it exists, is necessarily smooth of class at least $\mathcal{C}^r$.

References

[6] F. Deahna. "Ueber die Bedingungen der Integrabilität linearer Differentialgleichungen erster Ordnung zwischen einer beliebigen Anzahl veränderlicher Größen". In: *Journal für die reine und angewandte Mathematik* 20 (1840), pp. 340–349.

[7] A. Clebsch. "Ueber die simultane Integration linearer partieller Differentialgleichungen". In: *Journal für die reine und angewandte Mathematik* 65 (1866), pp. 257–268.

[8] F. G. Frobenius. "Ueber das Pfaffsche Problem". In: *Journal für die reine und angewandte Mathematik* 82 (1877). Reprinted in *Gesammelte Abhandlungen*, Vol. I, pp. 286–301, pp. 267–282.

[9] C. Chevalley. *Theory of Lie Groups.* See pp. 85–98. Princeton: Princeton University Press, 1946.

[10] É. Cartan. *Leçons sur la géométrie des espaces de Riemann.* See pp. 367–371. Paris: Gauthier-Villars, 1951.

[11] R. S. Palais. *A Global Formulation of the Lie Theory of Transformation Groups.* Memoirs of the American Mathematical Society 22. Providence, RI: American Mathematical Society, 1957.

[12] S. Lang. *Introduction to Differentiable Manifolds.* Chapter 6. New York: Wiley, 1962.

[13] S. Sternberg. *Lectures on Differential Geometry.* pp. 130–137. Englewood Cliffs, NJ: Prentice-Hall, 1964.

§3. Transversality

The first half of this section will consist of general remarks about transversality, and the second half will apply these remarks to foliation theory. All manifolds, maps, and foliations are to be smooth of class $\mathcal{C}^r$, where r is fixed, $1 \leqslant r \leqslant \infty$.

Let A, B and M be manifolds of dimension a, b and m respectively.

Definition 3.1. Two mappings $f : A \to M$ and $g : B \to M$ are **transverse** if, for every x in A and y in B with $f(x) = g(y)$, the following condition is satisfied. Setting $p = f(x) = g(y)$, the tangent vector space TM_p should be generated by the image of the two linear mappings

$$\mathrm{d}f_x : TA_x \to TM_p,$$
$$\mathrm{d}g_y : TB_y \to TM_p.$$

Alternatively, in terms of the dual linear mappings

$$\mathrm{d}f_x^* : TM_p^* \to TA_x^*,$$
$$\mathrm{d}g_y^* : TM_p^* \to TB_y^*,$$

the requirement is that the kernel of $\mathrm{d}f_x^*$ and the kernel of $\mathrm{d}g_y^*$ should be linearly independent.

Now consider the **product** $A \times_M B$ **over** M, consisting of all (x, y) in $A \times B$ with $f(x) = g(y)$. If f and g are transverse, it follows easily from the inverse function theorem that $A \times_M B$ is a smooth manifold of dimension $a + b - m$. (Compare Lang [12]). Note the commutative square of mappings

$$\begin{array}{ccc} A \times_M B & \longrightarrow & B \\ \downarrow & & \downarrow g \\ A & \xrightarrow{f} & M \end{array} .$$

Remark. IF $a + b < m$, then evidently the condition of transversality cannot be satisfied except in a trivial sense. Two maps $f : A \to M$ and $g : B \to M$ with $a + b < m$ are transverse only if the images $f(A)$ and $g(B)$ are disjoint.

Example. The most familiar kind of transversality occurs when A and B are submanifolds of M. In that case, $A \times_M B$ can be identified with the intersection $A \cap B$. (As examples, two distinct planes in 3-space always intersect transversally. But two intersecting lines in 3-space cannot intersect transversally.)

More generally, if $f : A \to M$ is arbitrary but B is a submanifold of M, then $A \times_M B$ can be identified with the inverse image $f^{-1}(B) \subset A$.

Note that a map $f : A \to M$ is transverse to every map from a manifold into M if and only if the derivative

$$\mathrm{d}f_x : TA_x \to TM_{f(x)}$$

is surjective for all x. In that case, the map f is called a **submersion**.

As an example, the projection map of a smooth fiber bundle is necessarily a submersion. If $f : E \to M$ is the projection map of a smooth fiber bundle, then the induced mapping

$$E \times_M B \to B$$

is called the **pullback**, or the **induced fiber bundle** over B.

Note that two maps f and g are transverse if and only if the product map

$$f \times g : A \times B \to M \times M$$

is transverse to the diagonal embedding

$$M \to M \times M.$$

In this form, the definition extends easily to any finite collection of mappings into M. For example, we define three maps

$$f : A \to M, \; g : B \to M, \; h : C \to M$$

to be **transverse** if and only if the product

$$f \times g \times h : A \times B \times C \to M \times M \times M$$

is transverse to the diagonal embedding

$$M \to M \times M \times M.$$

It then follows that $A \times_M B \times_M C$, the 3-fold product over M, is a smooth manifold of dimension $a + b + c - 2m$.

This definition can also be formulated in terms of the induced linear mappings of dual tangent spaces. The requirement is that, for each $x \in A$, $y \in B$, $z \in C$, and $p \in M$ with

$$f(x) = g(y) = h(z) = p,$$

the kernels of the three mappings

$$TM_p^* \to TA_x^*,\ TM_p^* \to TB_y^*,\ TM_p^* \to TC_z^*$$

should be linearly independent.

Using the latter formulation, it is not difficult to check that transversality obeys a form of associative law. Consider for example three maps f, g, h into M, where f is assumed to be transverse to g. Then the three maps f, g, h are transverse if and only if $f \times_M g$ is transverse to h. Details on all of these matters will be left to the reader.

Now suppose that M is a foliated manifold. Let

$$f : L \to M$$

be a smooth codimension q foliation.

Lemma 3.2. Every mapping $g : A \to M$ which is transverse to f give rise to a smooth codimension q foliation

$$L \times_M A \to A$$

of the manifold A.

This foliation $L \times_M A \to A$ is called the **pullback**, or the **induced foliation** of A.

Example. If A is a submanifold of M, then $L \times_M A$ can of course be identified with the submanifold $f^{-1}(A) \subset L$. The condition of transversality means intuitively that each leaf of the foliation f intersects A transversally. The components of these intersections form the leaves of the induced foliation $f^{-1}(A) \to A$.

Here is a quite different example. Let $g : A \to M$ be a submersion, and let $f : L \to M$ be the pointwise foliation of Example 1.2. Then, intuitively, the induced foliation $L \times_M A \to A$ is the foliation of A into fibers g^{-1}(constant). The codimension q, in this case, is equal to the dimension of M.

Proof of Lemma 3.2. Clearly $L \times_M A$ is a smooth manifold of dimension $a - q$ which maps bijectively to A. The proof of local flatness will be divided into four steps.
Case 1. Suppose that A is a convex open subset of the Euclidean space $\mathbb{R}^a$, that M is an open subset of $\mathbb{R}^m$, and that $g : A \to M$ is a linear mapping of rank m. Suppose further that $f : L \to M$ is the pointwise foliation of Example 1.2.

Inspection shows that each component of $L \times_M A$ maps diffeomorphically onto an $(a - m)$-plane of the form g^{-1}(constant) in A. Since these planes are mutually parallel, this completes the proof in Case 1.
Case 2. Now let A and M be arbitrary, but continue to assume that $f : L \to M$ is the pointwise foliation of Example 1.2. Evidently a mapping $g : A \to M$ is transverse to f only if g is a submersion. But, if g is a submersion, then given any point x of A we can choose a coordinate chart

$$h : U \to h(U) \subset \mathbb{R}^a$$

about x, and a coordinate chart

$$h' : V \to h'(V) \subset \mathbb{R}^m$$

about $g(x)$, with $g(U) \subset V$, so that the composition

$$h' \circ g \circ h^{-1} : h(U) \to h'(V)$$

is linear. (See for example the page 20 of [12]). Further, we can assume that the open set $h(U)$ is convex. The proof now proceeds as in Case 1.

Case 3. Suppose that the given foliation $f : L \to M$ is induced from the pointwise foliation $L' \to Q$ of some manifold Q by means of a submersion $s : M \to Q$. Since g is assumed transverse to f, an easy argument shows the composition

$$s \circ g : A \to Q$$

is again a submersion.

We must study the induced map $L \times_M A \to A$. Substituting $L = L' \times_Q M$, and noting that

$$(L' \times_Q M) \times_M A = L' \times_Q (M \times_M A) = L' \times_Q A,$$

it becomes clear that this induced map is isomorphic to the projection

$$L' \times_Q A \to A,$$

which is a foliation by Case 2.

General Case. Locally, over a sufficiently small open set $U \subset M$, any codimension q foliation f is induced from the pointwise foliation of $\mathbb{R}^q$ by means of a suitably chosen submersion

$$s : U \to \mathbb{R}^q.$$

Consider the corresponding open subsets

$$f^{-1}(U) \subset L, g^{-1}(U) \subset A,$$

and

$$f^{-1}(U) \times_U g^{-1}(U) = L \times_M g^{-1}(U) \subset L \times_M A.$$

According to Case 3, the induced mapping

$$f^{-1}(U) \times_U g^{-1}(U) \to g^{-1}(U)$$

is indeed a foliation of $g^{-1}(U)$. This proves local flatness, and completes the proof of Lemma 3.2. □

Here is a slightly different use of transversality in foliation theory. Consider a collection of smooth foliations

$$f_1 : L_1 \to M, \ldots, f_k : L_k \to M$$

of a single manifold M, with codimensions equal to $q_1, \ldots, q_k$ respectively.

Lemma 3.3. If the mappings $f_1, \ldots, f_k$ are transverse, then the product mapping

$$f_1 \times_M \cdots \times_M f_k : L_1 \times_M \cdots \times_M L_k \to M$$

is a foliation of M with codimension equal to $q_1 + \cdots + q_k$.

The proof will be left to the reader.

As an example to illustrate this phenomenon, the torus $S^1 \times S^1 \times S^1$ clearly possesses three mutually transverse codimension 1 foliations.

Startlingly enough, the sphere S^3 also possesses three mutually transverse codimension 1 foliations. These foliations, which are extremely difficult to visualize, are described in the following paper.

References

[14] D. Tischler. “Totally Parallelizable 3-Manifolds”. In: *Topological Dynamics*. Ed. by Joseph Auslander and Walter H. Gottschalk. New York: W. A. Benjamin, 1968, pp. 471–492.

§4. The Phillips-Gromov Theorem

In this section, all manifolds are to be paracompact, and smooth of class at least $\mathcal{C}^1$. The tangent vector bundle of a manifold M will be denoted by the symbol τM, while the total space of this tangent bundle will be denoted by TM.

Consider first two manifolds A and M of dimensions $a \geqslant m$. Let

$$\mathrm{Sub}(A, M)$$

be the space of all $\mathcal{C}^1$-submanifolds from A to M. We give the space the $\mathcal{C}^1$ compact open topology.

(By definition, the $\mathcal{C}^1$ **compact open topology** on a space of $\mathcal{C}^1$ mappings $g : A \to M$ is induced from the compact open topology on the space $\mathrm{Map}(TA, TM)$ of continuous mappings form TA to TM by means of the natural embedding $g \mapsto \mathrm{d}g \in \mathrm{Map}(TA, TM)$.)

Let $\mathrm{Epi}(\tau A, \tau M)$ denote the space of all mappings from the tangent manifold TA to the tangent manifold TM which carry every fiber TA_x linearly and surjectively to some fiber TM_y. We give this space the compact open topology.

Evidently the correspondence $g \mapsto \mathrm{d}g$ gives rise to a canonical embedding

$$\mathrm{d} : \mathrm{Sub}(A, M) \to \mathrm{Epi}(\tau A, \tau M).$$

Theorem (Phillips Submersion Theorem)**.** If the manifold A has no compact component, then the embedding

$$\mathrm{d} : \mathrm{Sub}(A, M) \to \mathrm{Epi}(\tau A, \tau M)$$

is a weak homotopy equivalence.

In particular, d induces isomorphisms of homotopy groups, and of singular homology groups, in all dimensions.

For the proof we refer to A. Phillips [16]. This theorem was motivated by an analogous theorem for immersions which had been proved earlier by Smale and Hirsch.

Note that the non-compactness restriction on A is essential. For example, if A is compact and parallelizable, then the space $\mathrm{Sub}(A, \mathbb{R}^m)$ is vacuous, but $\mathrm{Epi}(\tau A, \tau \mathbb{R}^m)$ is non-vacuous. (This is in contrast with the Smale-Hirsch theory of immersions of a manifold into a higher dimensional manifold, which does not require any non-compactness hypothesis).

The Submersion Theorem has an important application in foliation theory. Let Φ be a continuous field of tangent l-planes on the manifold A. We would like to be able to decide whether or not some homotopic l-plane field is tangent to a foliation of A.

Setting $q = a - l$, let $\tau A/\Phi$ be the quotient q-dimensional vector bundle over A, in which two vectors of TA_x are identified if and only if they are congruent modulo Φ_x.

Corollary 4.1. Suppose again that no compact of A is compact. If the vector bundle $\tau A/\Phi$ over A is trivial, then some l-plane field homotopic to Φ is tangent to an l-dimensional foliation of A.

Proof. If $\tau A/\Phi$ is trivial, then evidently there exists a fiberwise linear and surjective map

$$h : TA \to T\mathbb{R}^q$$

with "kernel" equal to Φ. Applying the Submersion Theorem, we see that h is homotopic to $\mathrm{d}g$ for some submersion

$$g : A \to \mathbb{R}^q.$$

The pointwise foliation of $\mathbb{R}^q$ now induces the required foliation of A. □

Remark. One outstanding problem of foliation theory is to know whether this corollary remains true for compact manifolds. John wood has shown that it is true for 3-manifolds, but very little is known in higher dimensions.

Remark. The proof above yields only a $\mathcal{C}^1$-foliation of A. However, if A is a $\mathcal{C}^r$-manifold, with $r > 1$, then we can easily approximate g by a $\mathcal{C}^r$-submersion, and thus obtain a $\mathcal{C}^r$-foliation.

Now suppose that we start with a smoothly foliated manifold

$$f : L \to M$$

of codimension q. Let

$$\mathrm{Trans}(A, M; f)$$

be the space of all $\mathcal{C}^1$-mappings from A to M which are transverse to f. We give this space the $\mathcal{C}^1$ compact open topology.

Let Φ be the field of l-planes tangent to the given foliation f, and let $\tau M/\Phi$ be the quotient q-plane bundle over M. For any $\mathcal{C}^1$-mapping $g : A \to M$ we consider the induced mapping $\mathrm{d}g : TA \to TM$, followed by the projection

$$p : TM \to TM/\Phi.$$

If g is transverse to f, then evidently the composition

$$p \circ \mathrm{d}g : TA \to TM/\Phi$$

carries every fiber TA_x surjectively onto some fiber TM_y/Φ_y.

Theorem (Phillips-Gromov Theorem)**.** If A has no compact component, then this construction yields a weak homotopy equivalence $p \circ d$ from the space

$$\mathrm{Trans}(A, M; f)$$

of mappings transverse to the foliation to the space

$$\mathrm{Epi}(\tau A, \tau M/\Phi)$$

of surjective bundle mappings.

As an example, if $f : L \to M$ is the pointwise foliation of Example 1.2, then this theorem evidently reduces to the Submersion Theorem stated earlier.

For the proof, we refer to Phillips [18]. Alternatively, Phillips points out that this result follows from a more general theorem proved by Gromov [15].

Applications of the Phillips-Gromov theorem will be given in subsequent sections. For the moment, we note only the following.

Corollary 4.2 (Phillips)**.** If A has no compact component, then every tangent plane field Φ of codimension 1 on A is homotopic to the field of planes tangent to a codimension 1 foliation of A.

This result is best possible, in a sense, since Bott has given counter-examples in codimension 2.

Proof of Corollary 4.2. Let E be the total space of the twisted line bundle over the real projective space P^n. For example, E can be obtained from the product $S^n \times \mathbb{R}$ by identifying each pair (x, t) with $(-x, -t)$. Note that E has a canonical codimension 1 foliation, corresponding to the foliation of $S^n \times \mathbb{R}$ into spheres $S^n \times$ (constant). (Compare Figure 4, for the case $n = 1$).

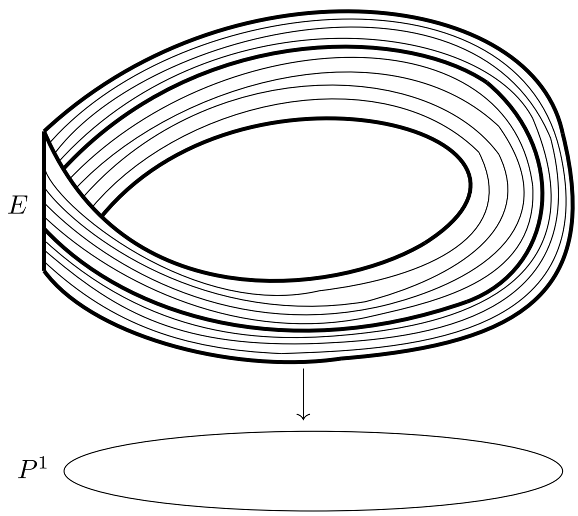

Figure 4. Foliation of the Twisted Line Bundle over P^1.

Let Ψ be the field of n-planes $\Psi_y \subset TE_y$ tangent to this foliation. Note that the manifold E contains P^n as a leaf. Correspondingly the line bundle $\tau E/\Psi$ over E contains a twisted line bundle over the subspace $P^n \subset E$.

Since the twisted line bundle over P^n is universal, there exists a non-degenerate bundle map

$$\tau A/\Phi \to \tau E/\Psi,$$

providing that $a \leqslant n$. Hence the space

$$\mathrm{Epi}(\tau A, \tau E/\Psi)$$

contains a map h with "kernel" equal to Φ. By the theorem, this h is homotopic to $p \circ \mathrm{d}g$ for some mapping

$$g : A \to E$$

transverse to the foliation of E. Clearly g induces the required foliation of A. □

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§5. Foliated Vector Bundles

Let ξ be a smooth vector bundle with base space $B = B(\xi)$, total space $E = E(\xi)$, and projection map

$$p : E \to B.$$

(The word "smooth" is again used to mean smooth of class $\mathcal{C}^r$, with $r \geqslant 1$ fixed). We will sometimes write ξ^n for ξ, where n is the dimension of fibers $\xi_x = p^{-1}(x)$.

Definition 5.1. A **foliation** of the vector bundle ξ will mean a smooth foliation f of the total space E which is transverse to the inclusion map $\xi_x \to E$ for every fiber ξ_x. (Compare Figure 5).

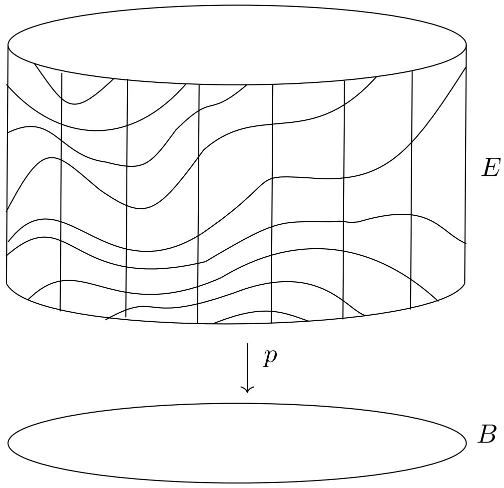

Figure 5

In other words, the foliation $f : L \to E$ is required to be transverse to the foliation of E by fibers (induced from the pointwise foliation of B). This transversality condition is clearly equivalent to the requirement that the composition $p \circ f$ should be a submersion of the leaf manifold L in B.

Note that the codimension q of such a foliation is related to the fiber dimension n by the inequality $0 < q \leqslant n$. We will be particularly interested in the case $q = n$. (Compare 5.8).

Example. The trivial n-plane bundle, with total space $B \times \mathbb{R}^n$, has a codimension n foliation induced from the pointwise foliation of $\mathbb{R}^n$. In §6, we will see that the tangent bundle of any $\mathcal{C}^\infty$-manifold of dimension n possesses a smooth codimension n foliation. Here is another example.

Lemma 5.2. If a smooth n-plane bundle has discrete structural group, then it possesses a codimension n foliation.

(Compare Figure 4). For the hypothesis of discrete structural group means precisely that the base space is covered by open sets U_α, with $p^{-1}(U_\alpha)$ diffeomorphic to $U_\alpha \times \mathbb{R}^n$, such that the foliation of $U_\alpha \times \mathbb{R}^n$ into horizontal slices $U_\alpha \times$ (constant) is preserved by the coordinate transformations

$$(U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n.$$

Remark 5.3. Not every n-plane bundle admits a codimension n foliation. Bott has proved that at least the following necessary condition must be satisfied in the $\mathcal{C}^\infty$ case. All products of rational Pontryagin classes

$$p_{i_1}(\xi) \cdots p_{i_k}(\xi) \in H^{4(i_1+\cdots+i_k)}(B; Q)$$

must vanish, providing that the dimension $4(i_1 + \cdots + i_k)$ is greater than $2n$. The simplest example of a vector bundle which does not satisfy Bott's condition is provided by a non-trivial 2-plane bundle over complex projective 4-space.

A major convenience in working with foliated vector bundles is that they behave nicely with respect to mappings. Consider for example two smooth vector bundles ξ and η, and a foliation f of η.

Lemma 5.4. If a smooth map $g : E(\xi) \to E(\eta)$ carries every fiber ξ_x linearly and surjectively to some fiber η_y, then g is transverse to the foliation f, and induces a foliation of the vector bundle ξ.

Proof. Since the restriction of g to any fiber ξ_x is transverse to f, it follows a fortiori that g is transverse to f, so that g does induce a foliation f' of $E(\xi)$. Furthermore, since the composition

$$\xi_x \to E(\xi) \to E(\eta)$$

is transverse to f, it follows easily that the inclusion $\xi_x \to E(\xi)$ is transverse to f'. □

Example 5.5. Suppose that ξ splits as a Whitney sum $\eta \oplus \zeta$. If η admits a codimension q foliation, then so does ξ. For the natural projection $E(\eta \oplus \zeta) \to E(\eta)$ certainly satisfies the hypothesis of Lemma 5.4.

Example 5.6. Given any smooth manifold A, and any smooth map

$$h : A \to B(\eta)$$

we can form the induced fiber bundle $h^*(\eta)$ over A, with total space

$$E(h^*(\eta)) = E(\eta) \times_{B(\eta)} A.$$

Evidently the canonical map

$$E(h^*(\eta)) \to E(\eta)$$

satisfies the hypothesis of Lemma 5.4. Hence any codimension q foliation of η induces a codimension q foliation of $h^*(\eta)$.

Next let us look at foliations which are given only on a small neighborhood of the zero cross-section.

Lemma 5.7. Let U be an open neighborhood of the zero-section in $E(\xi)$, and let f be a smooth foliation of U which is transverse to every fiber $\xi_x \cap U$. If $B(\xi)$ is paracompact, then there exists a foliation f' of ξ which coincides with f throughout some (possibly smaller) neighborhood of the zero-section.

This result can be expressed more sufficiently by saying that every "microfoliation" of ξ extends to a foliation of ξ. Here a **microfoliation** means an equivalence class of foliation, each defined on a neighborhood of the zero-section in $E(\xi)$, and each transverse to every fiber, where two such are defined to be equivalent if they coincide throughout some sufficiently small neighborhood of the zero-section.

Proof of Lemma 5.7. Since $B(\xi)$ is paracompact, we can first choose a smooth Euclidean metric for the vector bundle ξ, and then choose a smooth function $\varepsilon(x) > 0$ on $B(\xi)$ so that the open ball of radius $\varepsilon(x)$ in each fiber ξ_x is contained in U.

Let $\lambda : [0, \infty) \to [0, 1)$ be a smooth function of a real variable so that

$$\lambda(t) = t \qquad \text{for } t \leqslant \frac{1}{2},$$
$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} > 0 \quad \text{everywhere.}$$

Then a smooth embedding

$$h : E(\xi) \to E(\xi)$$

is defined as follows. For each vector v in ξ_x let

$$h(v) = vh(\|v\|)/\|v\|.$$

Evidently h maps each ξ_x diffeomorphically into the unit ball in ξ_x, leaving the ball of radius $1/2$ pointwise fixed.

Setting $h'(v) = \varepsilon(x) h\left(\frac{v}{\varepsilon(x)}\right)$, where $x = p(v)$, it follows that h' embeds $E(\xi)$ smoothly into U, and that h' restricts to the identity on the open set consisting of all v with $\|v\| < \frac{\varepsilon(x)}{2}$.

Evidently this embedding h' is transverse to the given foliation f of U, and hence induces a foliation f' of ξ. This completes the proof. □

Here is one application of Lemma 5.7. Let $\xi = \xi^n$ be a smooth vector bundle over a paracompact base space.

Corollary 5.8. The vector bundle ξ admits a codimension q foliation if and only if it splits as a Whitney sum $\eta \oplus \zeta$ where η is a q-plane bundle which admits a codimension q foliation.

Proof. By Example 5.5, any foliation of such a Whitney summand η^q gives rise to a foliation of ξ^n.

Conversely, given a codimension q foliation f of ξ^n which is smooth of class $\mathcal{C}^r$, we can restrict to any fiber $\xi^n_x = p^{-1}(x)$ to obtain a codimension q foliation of the fiber. Let φ^{n-q}_x be the sub vector space of ξ^n_x which is tangent to the leaf through the zero vector of this fiber. Then evidently these spaces φ^{n-q}_x form the fibers of a vector bundle

$$\varphi^{n-q} \subset \xi^n.$$

But unfortunately, this sub bundle is smooth only of class $\mathcal{C}^{r-1}$.

Choose a complementary sub bundle $\eta^q \subset \xi^n$, which is smooth of class $\mathcal{C}^r$, in such a way that ξ^n splits as the Whitney sum $\eta^q \oplus \varphi^{n-q}$. (To find such an η^q one needs to apply the approximation theorem which says that any section of a $\mathcal{C}^r$-fiber bundle over a paracompact base can be approximated arbitrarily closely by a $\mathcal{C}^r$-section. This theorem is applied to the Grassmann bundle over B whose fiber is the space of q-planes in ξ^n_x.)

Evidently the canonical embedding $E(\eta^q) \to E(\xi^n)$ is transverse to the given foliation f, at least if we restrict attention to a small neighborhood of the zero-section in $E(\eta^q)$. Hence this embedding induces a smooth microfoliation of η^q. Applying Lemma 5.7, it follows that η^q possesses a smooth codimension q foliation.

Clearly ξ^n splits smoothly as the Whitney sum $\eta^q \oplus \zeta^{n-q}$, where ζ^{n-q}_x is the orthogonal complement of η^q_x in ξ^n_x with respect to some smooth Euclidean metric. This completes the proof. □

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§6. Foliation of the Tangent Bundle

In this section we apply the Phillips-Gromov theorem to study the relationship between foliations of a manifold and foliations of its tangent bundle. Throughout the section M will denote a paracompact manifold which is smooth of class $\mathcal{C}^\infty$.

Theorem 6.1. Assume that M has no compact component. Then M possesses a smooth codimension q foliation of class $\mathcal{C}^r$, $r \geqslant 1$, if and only if the tangent bundle τM possesses a smooth codimension q foliation of class $\mathcal{C}^r$.

Thus the problem of foliating M is equivalent to the problem of foliating τM. The latter problem is easier to work with, since it is more amenable to the usual technique of homotopy theory.

Remark. A fundamental unsolved problem is the following. Does Theorem 6.1 remain true for compact manifolds? As an example, the tangent bundle of any odd dimensional sphere splits off a line bundle, and hence admits a codimension 1 foliation by Example 5.5. But no sphere of dimension > 3 is known to possess a codimension 1 foliation.

Here is a useful variant of 6.1. Let Ψ be a $\mathcal{C}^r$-smooth field of tangent $(m-q)$-planes on the manifold M. Again assume that M has no compact component.

Theorem 6.2. The manifold M possesses a smooth codimension q foliation whose field of tangent planes is homotopic to Ψ if and only if the q-plane bundle $\tau M/\Psi$ admits a smooth codimension q foliation.

Proof of Theorem 6.1. For the first half of the argument, the non-compactness hypothesis will not be needed.

Choose a $\mathcal{C}^\infty$ Riemannian metric on M, and choose a smooth function $\varepsilon(x) > 0$ on M, so that the exponential map carries the open ball of radius $\varepsilon(x)$ in each fiber TM_x diffeomorphically to an open subset of M. (Compare the page 147 of [23]). Let $U \subset TM$ be the open set consisting of all tangent vectors v with $\|v\| < \varepsilon(p(v))$.

Thus the exponential map is defined throughout U. Evidently any foliation f of M induces a foliation, which we will denote by $\exp^* f$, of this open set U. For each fiber TM_x, since the foliation f is transverse to the composition

$$TM_x \cap U \subset U \xrightarrow{\exp} M,$$

it follows that the induced foliation $\exp^* f$ is transverse to the inclusion

$$TM_x \cap U \to U.$$

Thus we have constructed a microfoliation of the tangent bundle τM. Applying 5.7, this microfoliation extends to a foliation of τM. This proves half of Theorem 6.1.

Before proving the other half, it will be convenient to state the Phillips-Gromov theorem in a form slightly different from that of §4. Let M, E be smooth manifolds, let f be a foliation of E, and let

$$\Phi_y \subset TE_y$$

be the associated field of tangent planes on E. Then

$$\mathrm{trans}(\tau M, \tau E; \Phi)$$

will denote the space of all mappings from TM to TE which carry each fiber TM_x linearly to some fiber TE_y, this linear mapping being transverse to the inclusion $\Phi_y \to TE_y$. We give this space the compact open topology. Evidently the natural projection

$$\mathrm{trans}(\tau M, \tau E; \Phi) \to \mathrm{Epi}(\tau M, \tau E/\Phi)$$

is a weak homotopy equivalence. So we can restate the Phillips-Gromov theorem as follows.

Theorem. The mapping

$$\mathrm{d} : \mathrm{Trans}(M, E; f) \to \mathrm{trans}(\tau M, \tau E; \Phi)$$

is a weak homotopy equivalence, providing that M has no compact component.

Now let E be the total space of a smooth vector bundle ξ over M. A canonical "vertical embedding"

$$V : E \to TE$$

is defined as follows. Let $z : M \to E$ be the zero cross-section. For each $x \in M$ the inclusion $i : \xi_x \subset E$ induces a linear embedding

$$\mathrm{d}i : T(\xi_x)_{z(x)} \to TE_{z(x)}.$$

But the tangent space $T(\xi_x)_{z(x)}$ is canonically isomorphic to the vector space ξ_x itself. Thus we obtain a canonical embedding

$$\xi_x \to TE_{z(x)},$$

the image being the set of vectors which are "vertical" (i.e., tangent to the fiber). Combining these embeddings for all x, we obtain the required mapping V which carries each fiber of ξ linearly into a fiber of τE.

Let f be a foliation of ξ, and let $\Phi_y \subset TE_y$ be the associated field of tangent planes. Clearly the embedding $\xi_x \to TE_{z(x)}$ is transverse to $\Phi_{z(x)}$, so that V represents an element of the space $\mathrm{trans}(\xi, \tau E; \Phi)$.

Now suppose that ξ is the tangent bundle τM, so that $E = TM$. The Phillips-Gromov theorem asserts that the vertical embedding

$$V \in \mathrm{trans}(\tau M, \tau E; \Phi)$$

is homotopic, within the space $\mathrm{trans}(\tau M, \tau E; \Phi)$, to $\mathrm{d}g$ for some mapping

$$g : M \to E,$$

which we may assume (after suitable approximation) to be smooth of class $\mathcal{C}^r$. Since g is transverse to f, it induces the required foliation of M. This proves Theorem 6.1. □

Proof of Theorem 6.2. Given a smooth field Ψ of tangent planes on M, let ξ be the vector bundle $\tau M/\Psi$. Assume that this bundle ξ has a foliation f, with associated field of tangent planes $\Phi_y \subset TE(\xi)_y$.

For each $x \in M$, we can compose the projection map

$$TM_x \to \xi_x$$

with the vertical embedding

$$\xi_x \to TE(\xi)_{z(x)}$$

to obtain a linear map which is evidently transverse to $\Phi_{z(x)}$. Thus we have constructed a canonical element

$$V' \in \mathrm{trans}(\tau M, \tau E(\xi); \Phi).$$

Choose a smooth map

$$g : M \to E(\xi),$$

transverse to f, so that the image $\mathrm{d}g \in \mathrm{trans}(\tau M, \tau E(\xi); \Phi)$ belongs to the same path component as V'. Then evidently the tangent plane field associated with the induced foliation g^*f of M will be homotopic to Ψ. This proves that half of Theorem 6.2.

For the other half of proof, M is allowed to be compact. Given a $\mathcal{C}^r$-foliation of M, let Ψ' be the associated field of tangent planes. Then Ψ' is smooth only of class $\mathcal{C}^{r-1}$. But, as in Corollary 5.8, we can choose a complementary q-plane field η_x, which is smooth of class $\mathcal{C}^r$, so that

$$TM_x = \eta_x \oplus \Psi'_x.$$

Thinking of η_x as the fiber of a vector bundle $\eta \subset \tau M$, the exponential map restricted to η induces a microfoliation of η, which extends to a foliation of η.

Now let Ψ''_x be the orthogonal complement of η_x in TM_x. Evidently Ψ'' is a $\mathcal{C}^r$-smooth tangent plane field, homotopic to Ψ', with

$$\tau M/\Psi'' \cong \eta.$$

Finally let Ψ be any tangent plane field of class $\mathcal{C}^r$ which is continuously homotopic to Ψ', and hence is $\mathcal{C}^r$-homotopic to Ψ''. Using the covering homotopy theorem, we see that

$$\tau M/\Psi \cong \tau M/\Psi'' \cong \eta.$$

Since η has been foliated, this completes the proof. □

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§7. Integrable Homotopy

Let f_0 and f_1 be two different codimension q foliations of the same manifold M, both smooth of class $\mathcal{C}^r$.

Definition 7.1. The foliation f_0 and f_1 are **integrably homotopic** (or briefly 1-homotopic) of class $\mathcal{C}^r$ if there exists a $\mathcal{C}^r$-smooth codimension q foliation F of the product manifold $M \times \mathbb{R}$ which is transverse to every $M \times$ constant and which induces a foliation $\mathcal{C}^r$-isomorphic to f_0 on $M \times 0$, and a foliation $\mathcal{C}^r$-isomorphic to f_1 on $M \times 1$.

As usual, we will often leave out the $\mathcal{C}^r$'s, simply assuming that everything in sight is to be smooth of class $\mathcal{C}^r$.

Without loss of generality, we may assume that the foliation F, restricted to some neighborhood $M \times (-\varepsilon, \varepsilon)$ of $M \times 0$, is isomorphic to a product foliation.

$$f_0 \times \text{identity} : L_0 \times (-\varepsilon, \varepsilon) \to M \times (-\varepsilon, \varepsilon)$$

and similarly with a neighborhood of $M \times 1$. To achieve this, we need only to choose a smooth map $\lambda : \mathbb{R} \to \mathbb{R}$ which carries a neighborhood of 0 to 0 and a neighborhood of 1 to 1. Then

$$\text{identity} \times \lambda : M \times \mathbb{R} \to M \times \mathbb{R}$$

will certainly be transverse to f, and the induced foliation will have the required property.

Now, given an i-homotopy from f_0 to f_1, and an i-homotopy from f_1 to f_2, it is clear that we can piece together to obtain an i-homotopy from f_0 to f_2. (Compare Figure 6). **Thus integrable homotopy is an equivalence relation.**

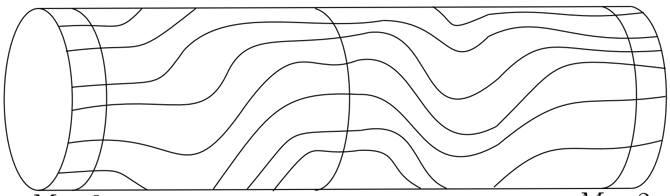

Figure 6. Piecing Together Integrable Homotopies.

Similarly, given two foliations f_0 and f_1 of a vector bundle ξ over M, we define f_0 and f_1 to be integrably **homotopic** if there exists a foliation of the induced vector bundle $\xi \times \mathbb{R}$ over $M \times \mathbb{R}$ which restricts to f_0 over $M \times 0$ and restricts to f_1 over $M \times 1$. Again this is an equivalence relation.

(Sometimes it is more convenient to use the unit interval $[0, 1]$ in place of the real line $\mathbb{R}$ as parameter manifold. To make sense of this, one must introduce the concept of a foliation, transverse to the boundary, of a smooth manifold with boundary. The resulting concept of 1-homotopy is completely equivalent to that defined above. Details will be left to the reader.)

It is now possible to sharpen some of the results of previous sections. Thus Lemma 5.7 becomes:

Lemma 7.2. Any microfoliation of ξ extends to a foliation of ξ which is unique up to i-homotopy.

Proof. Given two foliations f and f' of ξ which are isomorphic when restricted to a neighborhood U of the zero-section, we must prove that f is i-homotopic to f'. Choosing $\varepsilon(x)$ as in the proof of Lemma 5.7, recall that the mapping

$$h' : v \mapsto \varepsilon(x) h(v/\varepsilon(x)), \text{ for } v \in p^{-1}(x),$$

embeds $E(\xi)$ into U, and is the identity for $\|v\| \leqslant \varepsilon(x)/2$. Introducing a parameter $t \in \mathbb{R}$, we set

$$H(v,t) = h'(tv)/t \text{ for } t \neq 0,$$
$$H(v,0) = v.$$

Then evidently H is smooth as a function of both variables, mapping the total space $E(\xi) \times \mathbb{R}$ of $\xi \times \mathbb{R}$ onto $E(\xi)$. Lifting back the foliation f under H, we obtain a foliation $H^* f$ of $\xi \times \mathbb{R}$ which coincides with f on $\xi \times 0$, and with $h'^* f$ on $\xi \times 1$. Thus f is i-homotopic to $h'^* f$ and similarly f' is i-homotopic to $h'^* f'$. Since $h'^* f$ is clearly isomorphic to $h'^* f'$, this completes the proof. □

The results of §6 can be sharpened as follows. Let M be a paracompact $\mathcal{C}^\infty$ manifold.

Theorem 7.3. Every i-homotopy class of $\mathcal{C}^r$-foliations of M gives rise to a unique i-homotopy class of $\mathcal{C}^r$-foliations of the tangent bundle τM. If M has no compact component, then the converse is true: every i-homotopy class of $\mathcal{C}^r$-foliations of τM comes from one and only one i-homotopy class of $\mathcal{C}^r$-foliations of M.

This one-to-one correspondence is constructed precisely by the methods of §6. In particular, it preserves the codimension q.

The following special case is particularly striking.

Corollary 7.4. The tangent bundle of an m-dimensional manifold possesses one and only one i-homotopy class of codimension m foliations; providing that there is no compact component.

It may be conjectured that this corollary remains true for compact manifolds, although we will see in §8 that the theorem itself would be false for compact manifolds.

The proof of 7.3 will occupy the rest of §7. First some notation.

Let $\mathscr{F}(M)$ denote the set of all smooth i-homotopy classes of foliations of the manifold M (keeping the smoothness class $\mathcal{C}^r$ fixed). Similarly let $\mathscr{F}(\xi)$ be the set of all smooth i-homotopy classes of foliations of the vector bundle ξ. We will first construct a function

$$\varepsilon \exp^* : \mathscr{F}(M) \to \mathscr{F}(\tau M).$$

Starting with any foliation f of M, and choosing a Riemannian metric, the exponential map is defined and induces a microfoliation $\exp^* f$ of τM. Let the symbol ε stand for the operation of extending this microfoliation to a foliation. Then evidently $\varepsilon \exp^* f$ is a foliation of τM, well defined up to i-homotopy.

Clearly any i-homotopy of foliations of M gives rise, by this construction, to an i-homotopy of foliations of τM. Thus we have defined a function from the set $\mathscr{F}(M)$ of i-homotopy classes to the set $\mathscr{F}(\tau M)$.

This construction does not depend on the particular choice of Riemannian metric. For any two Riemannian metrics can be joined by a linear homotopy, which induces a homotopy of exponential maps, and hence an i-homotopy of foliations.

Now suppose that M has no compact component. Then a function

$$(\mathrm{d}^{-1}[V])^* : \mathscr{F}(\tau M) \to \mathscr{F}(M)$$

in the opposite direction can be constructed as follows. Given a foliation f of τM, let $\Phi_y \subset TE_y$ be the associated field of tangent planes, where $E = TM$, and let $[V]$ denote the path component of the vertical embedding in the space $\mathrm{trans}(\tau M, \tau E; \Phi)$ of fiberwise linear maps transverse to Φ. Choosing a smooth map

$$g : M \to E$$

so that $\mathrm{d}g$ belongs to this path component $[V]$, the induced foliation $g^* f$ of M is defined.

Note that the i-homotopy class of $g^* f$ does not depend on the particular choice of g. For if $\mathrm{d}g_0$ and $\mathrm{d}g_1$ belong to the same path component of $\mathrm{trans}(\tau M, \tau E; \Phi)$, then by the Phillips-Gromov theorem, g_0 and g_1 must belong to same path component of the space $\mathrm{Trans}(M, E; f)$. But any path from g_0 to g_1 in $\mathrm{Trans}(M, E; f)$ can be approximated by a path which is "smooth", in the sense that the associated map

$$M \times [0, 1] \to E$$

is $\mathcal{C}^r$-smooth. (Compare the section §4.5-4.6 of Munkres [24]). Given such a smooth path from g_0 to g_1 within $\mathrm{Trans}(M, E; f)$, it clearly follows that the induced foliations $g_0^* f$ and $g_1^* f$ are i-homotopic.

Note also that the i-homotopy class of the induced foliation $g^* f$ depends only on the i-homotopy class of the foliation f. In fact, any integrable homotopy between foliations f_0 and f_1 of τM is described by a foliation F of the product manifold $E \times \mathbb{R}$. Denote the associated tangent plane field by $\Phi'_{y,t} \subset T(E \times \mathbb{R})_{y,t}$. Since the vertical embedding of τM in $\tau E \times 0$ is evidently homotopic to the vertical embedding of τM in $\tau E \times 1$ within the space

$$\mathrm{trans}(\tau M, \tau(E \times \mathbb{R}); \Phi'),$$

it follows easily that the associated mappings

$$g_0 : M \to E \times 0,$$
$$g_1 : M \to E \times 1$$

are homotopic within the space

$$\mathrm{Trans}(M, E \times \mathbb{R}; F)$$

Hence $g_0^* f_0 \sim g_1^* f_1$. (Here the symbol $\sim$ stands for i-homotopy).

Thus we have constructed a well-defined function

$$\mathscr{F}(\tau M) \to \mathscr{F}(M),$$

to be denoted by the symbol $(\mathrm{d}^{-1}[V])^*$.

Lemma 7.5. The composition $\mathscr{F}(M) \xrightarrow{\varepsilon \exp^*} \mathscr{F}(\tau M) \xrightarrow{(\mathrm{d}^{-1}[V])^*} \mathscr{F}(M)$ is the identity map of $\mathscr{F}(M)$.

Proof. Start with a foliation f of M. Let $\Psi_x \subset TM_x$ be the associated field of tangent planes. Choose a neighborhood U of the zero-section in TM so that

$$\exp : U \to M$$

is defined, and let

$$\Phi_y \subset TU_y$$

be the field of tangent planes associated with the induced foliation $\exp^* f$.

Let $z : M \to U$ be the zero-section. Since the composition $\exp \circ z$ is the identity map of M, it is clear that z is transverse to the foliation $\exp^* f$, inducing back on M a foliation which is isomorphic to the original foliation f. Thus to prove Lemma 7.5, we need only show that the element

$$\mathrm{d}z \in \mathrm{trans}(\tau M, \tau U; \Phi)$$

belongs to the path component $[V]$ of the vertical embedding V. The tangent space $TU_{z(x)}$ clearly splits canonically as a direct sum

$$\mathrm{d}z_x(TM_x) \oplus T(TM_x)_{z(x)} \cong TM_x \oplus TM_x.$$

Using this direct sum decomposition, the two maps $\mathrm{d}z_x$ and V from TM_x to $TU_{z(x)}$ are evidently given by $w \mapsto w \oplus 0$ and $w \mapsto 0 \oplus w$ respectively. Each of these two maps is transverse to the subspace $\Phi_{z(x)}$.

In fact the subspace $\Phi_{z(x)}$ can be described explicitly as follows. Note that the homomorphism $\mathrm{d}\exp_{z(x)}$ carries each pair $w_1 \oplus w_2$ to the sum $w_1 + w_2 \in TM_x$. Hence $\Phi_{z(x)}$ consists of all pairs $w_1 \oplus w_2$ with $w_1 + w_2 \in \Psi_x$.

For each $t \in [0, 1]$, consider the homomorphism $w \mapsto tw \oplus (1-t)w$ from TM_x to $TU_{z(x)}$. Since this map is evidently transverse to the subspace $\Phi_{z(x)}$, it can be considered as a point in the space $\mathrm{trans}(\tau M, \tau U; \Phi)$. As t varies from 0 to 1, we thus obtain a path from V to $\mathrm{d}z$ within this space; completing the proof of Lemma 7.5. □

The proof in the other direction will be based on the following construction. Let ξ be a smooth vector bundle with paracompact base space, let N be a paracompact $\mathcal{C}^\infty$ manifold, and let f be a smooth foliation of N.

Definition. A smooth map from a neighborhood U of the zero-section in $E(\xi)$ to N is **fiberwise transverse** to f if its restriction to each fiber $\xi_x \cap U$ is transverse to the foliation f.

Clearly any such fiberwise transverse mapping $h : U \to N$ induces a microfoliations $h^* f$ of ξ, which extends to a foliation $\varepsilon h^* f$ of ξ.

Let $\Phi_y \subset TN_y$ be the tangent plane field associated with the foliation f. Then to each fiberwise transverse map h we associate an element

$$\mathrm{d}h \circ V \in \mathrm{trans}(\xi, \tau N; \Phi)$$

obtained by composing the vertical embedding of ξ in τU with the fiberwise linear map $\mathrm{d}h$.

Now consider two mappings h_0 and h_1 from U to N, both fiberwise transverse to f.

Lemma 7.6. If $\mathrm{d}h_0 \circ V$ and $\mathrm{d}h_1 \circ V$ belong to the same path component of the space $\mathrm{trans}(\xi, \tau N; \Phi)$, then the foliation $\varepsilon h_0^* f$ of ξ is i-homotopic to the foliation $\varepsilon h_1^* f$.

Proof. The proof is divided into three cases.
Case 1. First suppose that $\mathrm{d}h_0 \circ V = \mathrm{d}h_1 \circ V$. It follows that the two maps, as well as their first derivatives, coincide everywhere along the zero-section of ξ. Choosing a Riemannian metric on N, for every $x \in U$ close enough to the zero-section the two points $h_0(x)$ and $h_1(x)$ are joined by a unique minimal geodesic

$$t \mapsto h_t(x)$$

which depends smoothly on the endpoints. (See for example page 166 of [23]). Evidently we can choose a small neighborhood U' of the zero-section so that each h_t is defined on U'. Furthermore, since the first derivative of h_t coincides with the first derivative of h_t everywhere along the zero-section, the mappings h_t are fiberwise transverse to f throughout some smaller neighborhood U''. Thus we can lift back the foliation f to obtain a microfoliation, and hence a foliation, of the vector bundle $\xi \times [0,1]$. This proves that $\varepsilon h_0^* f \sim \varepsilon h_1^* f$.
Case 2. As an application of this argument, for any $h : U \to N$ which is fiberwise transverse to f, note that

$$\varepsilon h^* f \sim \varepsilon(\exp_N \circ \mathrm{d}h \circ V)^* f.$$

In fact, the map

$$\mathrm{d}h \circ V : E(\xi) \to TN$$

can be composed with the exponential map for N to yield a map

$$\exp_N \circ \mathrm{d}h \circ V : U' \to N$$

defined on a suitable neighborhood U' of the zero-section. Clearly this map coincides with h all along the zero-section, and the first derivatives also coincide. Thus the hypothesis of Case 1 is satisfied.
General Case. If $\mathrm{d}h_0 \circ V$ and $\mathrm{d}h_1 \circ V$ belong to the same path component of the space $\mathrm{trans}(\xi, \tau N; \Phi)$, then we can choose a smooth path between these points, corresponding to a smooth mapping

$$H : E(\xi) \times [0,1] \to TN.$$

Following H be the exponential map in N, we obtain

$$\exp_N \circ H : U' \times [0,1] \to N,$$

where U' is a suitable neighborhood of the zero-section. Thus the foliation f induces a microfoliation

$$(\exp_N \circ H)^* f$$

of the bundle $\xi \times [0,1]$, and this microfoliation extends to a foliation. This proves that

$$\varepsilon(\exp_N \circ \mathrm{d}h_0 \circ V)^* f \sim \varepsilon(\exp_N \circ \mathrm{d}h_1 \circ V)^* f.$$

Together with Case 2, it completes the proof of Lemma 7.6. □

Now we are ready for the final step.

Lemma 7.7. The composition $\mathscr{F}(\tau M) \xrightarrow{(\mathrm{d}^{-1}[V])^*} \mathscr{F}(M) \xrightarrow{\varepsilon \exp^*} \mathscr{F}(M)$ is the identity map of $\mathscr{F}(\tau M)$.

Proof. Starting with a smooth foliation f of τM, and setting $E = TM$, we first choose a smooth map $g : M \to E$, transverse to f, so that $\mathrm{d}g$ is homotopic to V within the space $\mathrm{trans}(\tau M, \tau E; \Phi)$. Then the foliation f lifts back to a foliation $g^* f$ of M, which lifts back to a microfoliation $\exp^* g^* f$ of τM. Extending to a foliation $f' = \varepsilon \exp^* g^* f$, we must prove that $f' \sim f$.

We will apply Lemma 7.6 to the vector bundle τM and the foliated manifold E. As maps

$$h_0, h_1 : U \to E$$

we take the inclusion map i, and the composition $g \circ \exp$ of the maps

$$U \xrightarrow{\exp} M \xrightarrow{g} E.$$

Evidently the foliations of τM associated with these two mappings are just f and f' respectively.

Clearly $\mathrm{d}i \circ V \in \mathrm{trans}(\tau M, \tau E; \Phi)$ is equal to the vertical embedding V, and $\mathrm{d}(g \circ \exp) \circ V \in \mathrm{trans}(\tau M, \tau E; \Phi)$ is equal to $\mathrm{d}g$. Since V and $\mathrm{d}g$ belong to the same path component by hypothesis, it follows from Lemma 7.6 that $f \sim f'$. This proves Lemma 7.7.

Evidently the two Lemmas 7.5 and 7.7 together prove Theorem 7.3. □

References

[24] J. R. Munkres. *Elementary Differential Topology.* Vol. 54. Annals of Mathematics Studies. Princeton University Press, 1966.

§8. An Example

This section will take a detailed look at the constant slope foliation of the torus $M = \mathbb{R}^2/\mathbb{Z}^2$. (Compare Example 1.4). Let f^s denote the codimension 1 foliation with constant slope $\dfrac{\mathrm{d}y}{\mathrm{d}x} = s$. We will prove the following two statements.

Assertion 8.1. If $s \neq s'$, then the foliation f^s is not $\mathcal{C}^r$ integrably homotopic to f^s for any $r \geqslant 2$.

(Presumably this statement is true for $r = 1$ also, but the present proof is not adequate for that case).

Assertion 8.2. The induced foliation $\exp^* f^s$ of the tangent bundle τM is $\mathcal{C}^\infty$ integrably homotopic to the induced foliation $\exp^* f^{s'}$ for all s and s'.

Thus if $\mathscr{F}(\ )$ stands for the set of all $\mathcal{C}^r$ integrable homotopy classes of foliations, where $r \geqslant 2$, then we have constructed uncountably many distinct elements in the set $\mathscr{F}(M)$ which all map to a single element under the function

$$\exp^* : \mathscr{F}(M) \to \mathscr{F}(\tau M)$$

This shows that Theorem 7.3 fails for compact manifolds, such as the torus.

The proof of Assertion 8.1 will be based on the following more general statement. Suppose that $r \geqslant 2$.

Theorem 8.3. If two smooth foliations of a compact manifold are $\mathcal{C}^r$ integrably homotopic, then there exists a $\mathcal{C}^{r-1}$-diffeomorphism of the manifold, $\mathcal{C}^{r-1}$-isotopic to the identity, which transforms one foliation to the other.

(Conversely, it is clear of course that if we are given any $\mathcal{C}^r$-isotopy $H : M \times \mathbb{R} \to M$, then any $\mathcal{C}^r$-foliation f of M pulls back to a $\mathcal{C}^r$ integrable homotopy $H^* f$ between foliations of M. This converse statement is true whether the manifold M is compact or not).

Proof of Theorem 8.3. Let F be a $\mathcal{C}^r$-foliation of $M \times \mathbb{R}$ which is transverse to every $M \times$ (constant), and let

$$\Phi_{x,t} \subset T(M \times \mathbb{R})_{x,t}$$

be the associated field of tangent planes. Clearly the field Φ is smooth class $\mathcal{C}^{r-1}$. Using a partition of unity, construct a $\mathcal{C}^{r-1}$ vector field $v(x,t)$ on $M \times \mathbb{R}$ so that

$$v(x,t) \in \Phi_{x,t},$$

and so that the derivative $\mathrm{d}\pi$ of the projection $\pi : M \times \mathbb{R} \to \mathbb{R}$ maps $v(x,t)$ to the constant vector field $\dfrac{\mathrm{d}}{\mathrm{d}t}$ on $\mathbb{R}$. Since M is compact, the vector field v generates a one parameter group of diffeomorphism

$$g_t : M \times \mathbb{R} \to M \times \mathbb{R},$$

where $g_t(x,u)$ is smooth of class $\mathcal{C}^{r-1}$ as a function of three variables. (Compare the page 12-14 of [23]. The condition $r \geqslant 2$ is needed here.) Note that g_t maps each $M \times c$ onto $M \times (c+t)$.

Evidently each g_t transforms the foliation F into itself: that is $g_t^* F \cong F$. So if F_i denotes the foliation induced by F on the submanifold $M \times i$, then

$$g_1^* F_1 \cong F_0$$

Now let f_i denote the foliation of M which corresponds to this foliation F_i of $M \times i$. Then clearly

$$h_1^* f_1 = f_0$$

where the diffeomorphism $h_t : M \to M$, defined by

$$(h_t(x), t) = g_t(x, 0),$$

is $\mathcal{C}^{r-1}$-isotopic to the identity map h_0. This completes the proof of Theorem 8.3. □

Again let f^s denote the slope s foliation of the torus.

Lemma 8.4. If $s \neq s'$, then no homeomorphism of the torus which is homotopic to the identity can transform f^s into $f^{s'}$.

Proof. To any foliation f of the torus M we will assign a topological invariant $\Lambda(f)$ which consists of a set of lines through the origin in the vector space $\pi_1 M \otimes \mathbb{R}$.

The given foliation f clearly pulls back to a foliation $\widetilde{f}$ of the universal covering manifold $\widetilde{M}$. Let K be a compact subset of $\widetilde{M}$ whose interior maps onto M, and let Σ_K be the set of all elements $\sigma \neq 1$ in the group $\pi_1 M$ of covering transformation such that both K and σK intersect some common leaf of the foliation $\widetilde{f}$. (Compare Figure 7). Then, embedding $\pi_1 M$ in the plane $\pi_1 M \otimes \mathbb{R}$, draw a line $\sigma \otimes \mathbb{R}$ from the origin through each point σ of Σ_K. Let $\Lambda_K(f)$ be the set consisting of all limit lines

$$\lim_{i\to\infty} \sigma_i \otimes \mathbb{R} \subset \pi_1 M \otimes \mathbb{R}$$

where $\{\sigma_i\}$ can be any infinite sequence of **distinct** elements of Σ_K for which this limit exists.

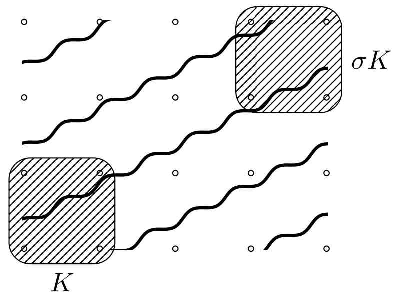

Figure 7

We claim that this set $\Lambda_K(f)$ of limit lines is independent of K. For if J is another compact subset of $\widetilde{M}$ whose interior maps onto M, then

$$J \subset \tau_1 K \cup \cdots \cup \tau_q K$$

for some finite collection of elements $\tau_1, \ldots, \tau_q \in \pi_1 M$. If $\sigma \in \Sigma_J$, then

$$\tau_\alpha^{-1} \sigma \tau_\beta \in \Sigma_K$$

for some α and β. It follows that

$$\Lambda_J(f) \subset \Lambda_K(f)$$

For if the line λ is the limit of a sequence $\sigma_i \otimes \mathbb{R}$ with $\sigma_i \in \Sigma_J$, then clearly λ is also the limit of a corresponding sequence $\tau_{\alpha_i}^{-1} \sigma_i \tau_{\beta_i} \otimes \mathbb{R}$.

This proves that $\Lambda(f)$ is well defined. It is evident from the construction that the set $\Lambda(f)$ is a topological invariant of the foliation f. In particular, suppose that two foliations f_1 and f_2 correspond under a homeomorphism $h : M \to M$. (This means that the following diagram is commutative.)

$$\begin{array}{ccc} L_1 & \cong & L_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ M & \xrightarrow{h} & M \end{array}$$

Then clearly the two sets $\Lambda(f_1)$ and $\Lambda(f_2)$ must correspond under the induced linear mapping

$$h_* : \pi_1 M \otimes \mathbb{R} \to \pi_1 M \otimes \mathbb{R}.$$

In particular, if h is homotopic to the identity, then $\Lambda(f_1) = \Lambda(f_2)$.

Finally, let us apply this construction to the constant slope foliation f^s. Evidently the invariant $\Lambda(f^s)$ consists of a single line of slope s in $\pi_1 M \otimes \mathbb{R}$. This completes the proof of Lemma 8.4. □

Combining Theorem 8.3 and Lemma 8.4, we clearly obtain the Assertion 8.1.

Proof of Assertion 8.2. The proof will be based on a suggestion of Haefliger, and is closely related to the page 468 of Reinhart [25].

In addition to the original coordinates x modulo 1 and y modulo 1 on the torus, we will use rotated coordinates a and b, where the rotation is chosen so as to carry the lines $y = sx +$constant to the horizontal lines $b =$ constant. The coordinate pair (a, b) associated with each point is well defined modulo some appropriate lattice of points in the plane.

Let α, β be corresponding coordinates in the tangent bundle, so that the projection from TM to M is given by

$$(a, b, \alpha, \beta) \mapsto (a, b),$$

and the exponential map by

$$(a, b, \alpha, \beta) \mapsto (a + \alpha, b + \beta).$$

Thus $\exp^*$ carries the foliation f^s of M, with typical leaf $b =$ constant, to the foliation of τM with typical leaf $b + \beta =$ constant.

Lemma 8.5. This foliation $\exp^* f^s$ of the vector bundle τM, with typical leaf

$$\{(a, b, \alpha, \beta) \mid b + \beta = \text{constant}\},$$

is integrably homotopic to the foliation F^s with typical leaf

$$\{(a, b, \alpha, \beta) \mid \beta = \text{constant}\}.$$

Proof. Consider an intermediate foliation F_1 of τM as follows. Each leaf of F_1 can be described locally as the graph of a function $\beta = \beta(a, b, \alpha)$ satisfying the differential equations $\dfrac{\partial\beta}{\partial a} = 0$, $\dfrac{\partial\beta}{\partial b} = g(\beta)$, $\dfrac{\partial\beta}{\partial\alpha} = 0$, where $g(\beta)$ is a $\mathcal{C}^\infty$ function satisfying the conditions

$$g(\beta) = -1 \quad \text{for } \beta \leqslant 1,$$
$$g(\beta) = 0 \quad \text{for } \beta \geqslant 2.$$

Locally, this foliation F_1 is induced from the foliation sketched in Figure 8 by means of the projection

$$(a, b, \alpha, \beta) \mapsto (b, \beta).$$

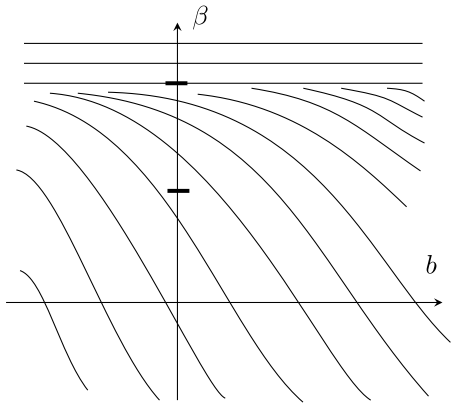

Figure 8

Evidently the two foliations $\exp^* f^s$ and F_1 are isomorphic throughout the neighborhood $\beta < 1$ of the zero-section in TM. Hence by Lemma 7.2, the foliation $\exp^* f^s$ is i-homotopic to F_1.

Now consider the smooth isotopy $h_t : TM \to TM$ defined by

$$h_t(a, b, \alpha, \beta) = (a, b, \alpha, \beta + t)$$

Evidently the foliation $h_3^* F_1$ is i-homotopic to F_1. But $h_3^* F_1$, coincides with F^s throughout a neighborhood of the zero-section. Thus $\exp^* f^s \sim F_1 \sim h_3^* F_1 \sim F^s$, which completes the proof of Lemma 8.5. □

Now, to prove Assertion 8.2, we need only show that $F^s \sim F^{s'}$. Switching back to the original coordinates x, y on M, and associated coordinates ξ, η on TM, clearly F^s has typical leaf $\eta = s\xi +$ constant.

Keeping x and y fixed, but rotating the ξ, η plane, we obtain a smooth isotopy of TM which transforms F^s to $F^{s'}$. This isotopy $TM \times \mathbb{R} \to TM$ induces the required i-homotopy, and completes the proof of Assertion 8.2. □

We conclude with an exercise for the reader. Define two foliations f_0 and f_1 of M to be **concordant** if there exists a foliation F of $M \times \mathbb{R}$ which is transverse to $M \times 0$ and $M \times 1$ (but not necessarily to every $M \times$ constant), and which induces a foliation isomorphic to f_0 (respectively f_1) on $M \times 0$ (respectively on $M \times 1$).

Problem (Haefliger)**.** Using Reinhart's technique, as in Figure 8, prove that the constant slope foliation f^s is concordant to $f^{s'}$.

References

[25] B. L. Reinhart. “Characteristic Numbers of Foliated Manifolds”. In: *Topology* 6 (1967), pp. 467–471.