

Category Theory

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Preface

Category theory, since its inception in the mid-20th century, has emerged as a profound unifying language for mathematics and beyond. What began as an abstract framework to formalize algebraic topology and homological algebra has grown into a discipline that reveals deep connections across disparate fields, from logic and computer science to physics and linguistics. This script, *Category Theory* by Paul Ziegler, offers an introduction to this transformative field, guiding readers through its foundational concepts, core principles, and applications with clarity and rigor.

At its heart, category theory is the study of mathematical structure through the lens of objects, arrows (morphisms), and their interactions. Unlike traditional approaches that focus on the internal details of mathematical objects, category theory emphasizes the relationships between structures and the patterns that arise from these interactions. This shift in perspective—from “what things are” to “how they behave and connect”—has proven remarkably powerful, enabling mathematicians to distill complex ideas into universal principles. The script opens with this philosophy, introducing categories as abstract environments where sets, groups, topological spaces, and other mathematical entities coexist under a common formalism.

The text is structured to build intuition progressively. Early chapters lay the groundwork with definitions of categories, functors, and natural transformations, illustrated through concrete examples: the category **Set** of sets and functions, **Grp** of groups and homomorphisms, poset categories, and monoid categories. These examples anchor abstract definitions in familiar territory, demonstrating how category theory reinterprets classical mathematics. The duality principle—the idea that every categorical concept has a dual, obtained by reversing arrows—is introduced early, foreshadowing its pervasive role in unifying constructions like products/coproducts and limits/colimits.

A central theme of the script is the interplay between universal properties and their realizations. Chapters on limits, pullbacks, and adjoints showcase how category theory provides a language to describe “optimal solutions” to structural problems, whether constructing products in **Set** or defining free categories from graphs. The Yoneda Lemma, a cornerstone of the theory, is presented not merely as a technical tool but as a profound statement about representation and duality, bridging local and global perspectives.

What sets this text apart is its balance of abstraction and application. While rigorously developing concepts like additive categories and biproducts, it consistently highlights connections to other disciplines. For instance, the category $\mathbf{Mat}(K)$ of matrices is shown to encode linear algebra in categorical terms, while examples from computer science and logic illustrate how categorical thinking underpins type systems, proofs, and computational processes. The script also touches on philosophical implications, such as viewing physical systems as categories where processes are morphisms—a perspective gaining traction in quantum theory.

Designed for graduate students and researchers in mathematics, computer science, or related fields, this script assumes familiarity with basic algebra and set theory but prioritizes conceptual clarity over encyclopedic detail. Exercises and proofs are interwoven to reinforce understanding, while diagrams and analogies demystify complex ideas. By the final chapters on adjoints and additive categories, readers will appreciate category theory not just as a technical toolkit but as a lens to reorganize and reimagine mathematical thought itself.

In an era where interdisciplinary research thrives, category theory offers a lingua franca for structural reasoning. This script invites readers to explore its elegance and power, equipping them to participate in the ongoing dialogue between abstraction and application that defines modern mathematics.

Xumin Liang

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Chapter 1

Categories

1.1 Introduction

This brief introduction is taken from Awodey's book.

What is category theory? As a first approximation, one could say that category theory is the mathematical study of (abstract) algebras of functions.

The historical development of the subject has been, very roughly, as follows:

- 1945: Eilenberg and Mac Lane's "General theory of natural equivalences" was the original paper, in which the theory was first formulated.
- Late 1940s: The main applications were originally in the fields of algebraic topology, particularly homology theory, and abstract algebra.
- 1950s: Grothendieck et al. began using category theory with great success in algebraic geometry.
- 1960s: Lawvere and others began applying categories to logic, revealing some deep and surprising connections.
- 1970s: Applications were already appearing in computer science, linguistics, cognitive science, philosophy, and many other areas.

One very striking thing about the field is that it has such wide-ranging applications. In fact, it turns out to be a kind of universal mathematical language like Set Theory. As a result of these various applications, category theory also tends to reveal certain connections between different fields like Logic and Geometry.

In fact, just as the idea of a topological space arose in connection with continuous functions, so also the notion of a category arose in order to define that of a functor, at least according to one of the inventors. The notion of a functor arose - so the story goes on - in order to define natural transformations. One might as well continue that natural transformations serve to define adjoints, so we have the following succession:

category \rightsquigarrow functor \rightsquigarrow natural transformation \rightsquigarrow adjunction.

Before getting down to business, let us ask why it should be that category theory has such far-reaching applications. Well, we said that it is the abstract theory of functions, so the answer is simply this:

Functions are everywhere!

And everywhere that functions are, there are categories. Indeed, the subject might better have been called abstract function theory, or, perhaps even better: archery.

1.2 Definition

Definition 1.1. A category \mathcal{C} consists of the following data:

- (1) **objects:** A, B, C, \dots (not necessarily sets). The class of objects of \mathcal{C} is denoted by $\text{Ob}(\mathcal{C})$.
- (2) **arrows** (or **morphisms**): f, g, h, \dots (not necessarily functions). The class of arrows of \mathcal{C} is denoted by $\text{Mor}(\mathcal{C})$. The class of arrows between two objects A, B of \mathcal{C} is denoted by $\text{Hom}_{\mathcal{C}}(A, B)$ or $\text{Mor}_{\mathcal{C}}(A, B)$.
- (3) For every arrow f there are objects $\text{dom}(f)$ and $\text{cod}(f)$ called the **domain** and the **codomain** of f respectively. We write $f : A \rightarrow B$, where $A = \text{dom}(f)$ and $B = \text{cod}(f)$.
- (4) For every arrows $f : A \rightarrow B$ and $g : B \rightarrow C$ there is an arrow denoted $g \circ f : A \rightarrow C$ and called the **composite** of f and g .
- (5) For every object A there is an arrow denoted $1_A : A \rightarrow A$ and called the **identity arrow**.

which satisfy the following axioms:

- (i) Associativity:

$$(h \circ g) \circ f = h \circ (g \circ f), \quad \forall f : A \rightarrow B, g : B \rightarrow C, h : C \rightarrow D$$

- (ii) Unit:

$$f \circ 1_A = f = 1_B \circ f, \quad \forall f : A \rightarrow B$$

If A, B are objects and $f : A \rightarrow B$ is an arrow in a category \mathcal{C} , then sometimes we simply denote $A, B \in \mathcal{C}$ and $f \in \mathcal{C}$ instead of $A, B \in \text{Ob}(\mathcal{C})$ and $f \in \text{Hom}_{\mathcal{C}}(A, B)$.

1.3 Examples

1.3.1 The Category Set

- objects: sets
- arrows: functions
- composition: composition of functions
- identity arrow: identity function

There are variations of this category obtained by restricting the sets and/or the functions, such as: the category \mathbf{Set}_{fin} of finite sets and functions, the category of sets and injective functions etc.

1.3.2 Categories of Structured Sets

- (1) The category of groups and group homomorphisms, denoted by **Grp**. The category of abelian groups and group homomorphisms, denoted by **Ab**.
- (2) The category of monoids and monoid homomorphisms, denoted by **Mon**.
- (3) The category of unitary rings and unitary ring homomorphisms, denoted by **Ring**. The category of commutative unitary rings and unitary ring homomorphisms, denoted by **CRing**.
- (4) The category of rings and ring homomorphisms, denoted by **Rng**. The category of commutative rings and ring homomorphisms, denoted by **CRng**.
- (5) The category of fields and field homomorphisms, denoted by **Field**.
- (6) The category of left vector spaces over a field K and K -linear maps, denoted by **Vect**(K). The category of left modules over a unitary ring R and R -module homomorphisms, denoted by **Mod**(R).

- (7) The category of graphs and graph homomorphisms, denoted by **Graph**.
- (8) The category of topological spaces and continuous maps, denoted by **Top**.
- (9) The category of real Banach spaces and linear contractions, denoted by **Ban**.
- (10) The category of differentiable (smooth) manifolds and differentiable (smooth) mappings, denoted by **Man**.
- (11) The category of preordered sets and monotone mappings, denoted by **Preord**. The category of posets (partially ordered sets) and monotone mappings, denoted by **Pos**.

Remark 1.1. All the above examples are concrete categories, which roughly speaking means that the objects are some sets and the arrows are some functions.

1.3.3 The Category Rel

- objects: sets
- arrows: relations $r = (A, B, R)$, where $R \subseteq A \times B$
- composition: composition of relations, defined for relations $r = (A, B, R)$ and $s = (B, C, S)$ as $s \circ r = (A, C, S \circ R)$, where

$$S \circ R = \{(a, c) \in A \times C \mid \exists b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\}$$

- identity arrow: for every set A , the identity arrow is the equality relation $\delta_A = (A, A, \Delta_A)$, where $\Delta_A = \{(a, a) \mid a \in A\}$.

1.3.4 Another Category of Finite Sets: $\text{Mat}(K)$

- Objects: finite sets (or simply natural numbers)
- arrows: for every finite sets A with $|A| = m \in \mathbb{N}$ and B with $|B| = n \in \mathbb{N}$, define an arrow $A \rightarrow B$ to be a matrix in $M_{m,n}(K)$ (where K is a fixed field).
- composition: multiplication of matrices
- identity arrow: identity matrix

1.3.5 Poset Categories

Given a poset (P, \leq) , we may construct an associated category called a **poset category**:

- Objects: the elements of P
- arrows: we say that there is an arrow between $a, b \in P$, and we write $a \rightarrow b$, if and only if $a \leq b$.
- composition: composition of arrows in the sense that $a \rightarrow b \rightarrow c$ if and only if $a \leq b \leq c$
- identity arrow: we have $a \rightarrow a$ for every object $a \in P$.

1.3.6 Monoid Categories

Given a monoid (M, \cdot) , we may construct an associated category:

- objects: the single object M
- arrows: the elements of M
- composition: the multiplication of the elements of M
- identity arrow: the identity element from the monoid

1.3.7 Finite Categories

- (1) The category **0**: it has no objects and no arrows.
- (2) The category **1**:

*

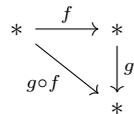
which has a single object $*$, the identity arrow as the single arrow and the composition is given by iteration of the identity arrow.

- (3) The category **2**:

$* \rightarrow *$

which has two objects, the identity arrows and one non-identity arrow between the two objects and the composition is given by two successive arrows.

- (4) The category **3**:



which has 3 objects, the identity arrows and the 3 depicted non-identity arrows between the objects and the composition is given by two successive arrows.

1.3.8 The Category **Cat**

Definition 1.2. A **covariant functor** (or **simply functor**) between two categories \mathcal{C} and \mathcal{D} is a mapping of objects of \mathcal{C} to objects of \mathcal{D} and of arrows of \mathcal{C} to arrows of \mathcal{D} , denoted by

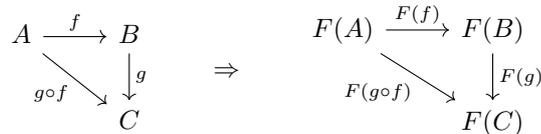
$$F : \mathcal{C} \rightarrow \mathcal{D}$$

satisfying the axioms:

- (i) For every $f : A \rightarrow B$ in \mathcal{C} , we have $F(f) : F(A) \rightarrow F(B)$ in \mathcal{D} .
- (ii) For every object A of \mathcal{C} , we have $F(1_A) = 1_{F(A)}$.
- (iii) For every composable pair of arrows $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathcal{C} , we have

$$F(g \circ f) = F(g) \circ F(f)$$

hence the commutativity of the following left diagram implies the commutativity of the right diagram:



For functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{E}$, one defines the composite functor $G \circ F : \mathcal{C} \rightarrow \mathcal{E}$ by

$$\begin{aligned}
 (G \circ F)(C) &= G(F(C)), & \text{for every object } C \text{ of } \mathcal{C} \\
 (G \circ F)(f) &= G(F(f)), & \text{for every arrow } f : A \rightarrow B \text{ of } \mathcal{C}
 \end{aligned}$$

The category **Cat**:

- objects: small categories (that is, categories \mathcal{C} such that $\text{Ob}(\mathcal{C})$ and $\text{Mor}(\mathcal{C})$ are sets)
- arrows: covariant functors
- composition: composition of functors
- identity arrow: identity functor $1_{\mathcal{C}} : \mathcal{C} \rightarrow \mathcal{C}$ for every category \mathcal{C} , defined by the identity on objects and on arrows.

1.3.9 A Category form Logic

Given a deductive system of logic, we may construct an associated category of proofs:

- objects: formulas φ, ψ, \dots
- arrows: implications $\varphi \rightarrow \psi$
- composition: successive implications $\varphi \rightarrow \psi \rightarrow \Delta$
- identity arrow: each formula implies itself

1.3.10 A Category from Computer Science

Given a functional programming language L , we may construct an associated category:

- objects: data types of L
- arrows: computable functions on L (“processes”)
- composition: successive computable functions $X \rightarrow Y \rightarrow Z$, where the output of the first arrow is the input of the second arrow
- identity arrow: “do nothing” procedure

1.3.11 A Category from Physics

- objects: physical system A, B, C, \dots
- arrows: physical processes which take a physical system of type of A into a physical system of type B
- composition: sequential composition of physical processes
- identity arrow: the physical process leaving the physical system invariant

1.4 Isomorphisms

Definition 1.3. In any category \mathcal{C} , an arrow $f : A \rightarrow B$ is called an **isomorphism** if there is an arrow $g : B \rightarrow A$ such that

$$g \circ f = 1_A \quad \text{and} \quad f \circ g = 1_B$$

Since inverses are unique, we write $g = f^{-1}$.

We say that A is **isomorphic** to B , written $A \cong B$ if there exists an isomorphism between them.

We recall the following famous theorem from Group Theory.

Theorem 1.2 (Cayley). Every group is isomorphic to a subgroup of a symmetric group.

Proof. Let (G, \cdot) be a group and consider the symmetric group

$$S_G = \{g : G \rightarrow G \mid g \text{ is bijective}\}.$$

For every $a \in G$, define

$$t_a : G \rightarrow G \text{ by } t_a(x) = ax, \quad \forall x \in G.$$

One proves that $t_a \in S_G$, that is t_a is bijective. We may now define

$$f : G \rightarrow S_G \text{ by } f(a) = t_a, \forall a \in G$$

One shows that f is an injective homomorphism. Then $\text{Ker } f = \{1\}$.

By the first isomorphism theorem, it follows that

$$G \cong G/\{1\} \cong G/\text{Ker } f \cong \text{Im } f.$$

But $\text{Im } f$ is a subgroup of S_G , so that we are done. Note that $\text{Im } f$ is sometimes called the Cayley representation of G . \square

Remark 1.3. Note the two different levels of isomorphisms that occur in the proof of Cayley’s theorem. There are bijective functions $g : G \rightarrow G$, which are isomorphisms in **Set**, and there is the isomorphism between G and $\text{Im } f$ in **Grp**. Cayley’s theorem says that any abstract group can be represented as a “concrete” one, that is, a subgroup of a symmetric group.

We may give the following category-theoretic analogue.

Theorem 1.4. Every category \mathcal{C} with a set of arrows is isomorphic to one in which the objects are sets and the arrows are functions.

Proof. Define the Cayley representation $\bar{\mathcal{C}}$ of \mathcal{C} , that is, the category corresponding to \mathcal{C} via the isomorphism, to be the following concrete category:

- objects: sets of the form $\bar{C} = \{f \in \mathcal{C} \mid \text{cod}(f) = C\}$ for objects C of \mathcal{C}
- arrows: functions $\bar{g} : \bar{C} \rightarrow \bar{D}$ for arrows $g : C \rightarrow D$ in \mathcal{C} , defined by $\bar{g}(f) = g \circ f$ for every $f : X \rightarrow C$ in \bar{C} .

One shows the required properties. □

1.5 Constructions on Categories

1.5.1 Product Category

Let \mathcal{C} and \mathcal{D} be categories. The **product category** $\mathcal{C} \times \mathcal{D}$ is defined as follows:

- objects: pairs (C, D) for some objects $C \in \mathcal{C}$ and $D \in \mathcal{D}$
- arrows: pairs (f, g) for some arrows $f \in \mathcal{C}$ and $g \in \mathcal{D}$
- composition: for every composable arrows $(f, g), (f', g') \in \mathcal{C} \times \mathcal{D}$, their composite is defined as

$$(f, g) \circ (f', g') = (f \circ f', g \circ g')$$

- identity arrow: for every object $(C, D) \in \mathcal{C} \times \mathcal{D}$, the identity arrow is $1_{(C,D)} = (1_C, 1_D)$.

Clearly, the construction may be generalized for a finite number of categories.

1.5.2 Opposite Category

Let \mathcal{C} be a category. The **opposite category** \mathcal{C}^{op} of \mathcal{C} is defined as follows:

- objects: the objects of \mathcal{C} . We denote by C^* the object C of \mathcal{C} viewed as an object of \mathcal{C}^{op} .
- arrows: the arrows of the form $f^* : B^* \rightarrow A^*$ for some arrow $f : A \rightarrow B$ in \mathcal{C} .
- composition: for every arrows $f^* : B^* \rightarrow A^*$ and $g^* : C^* \rightarrow B^*$ in \mathcal{C}^{op} , their composite is defined as

$$f^* \circ g^* = (g \circ f)^*$$

$$\begin{array}{ccc}
 A & \xrightarrow{f} & B \\
 & \searrow^{g \circ f} & \downarrow g \\
 & & C
 \end{array}
 \quad \Rightarrow \quad
 \begin{array}{ccc}
 A^* & \xleftarrow{f^*} & B^* \\
 & \swarrow^{(g \circ f)^*} & \uparrow g^* \\
 & & C^*
 \end{array}$$

- identity arrow: for every object $A^* \in \mathcal{C}^{\text{op}}$, the identity arrow is $1_{A^*} = (1_A)^*$.

1.5.3 Arrow Category

Let \mathcal{C} be a category. The **arrow category** $\mathcal{C}^{\rightarrow}$ of \mathcal{C} is defined as follows:

- objects: the arrows of \mathcal{C} .
- arrows: an arrow $g = (g_1, g_2) : f \rightarrow f'$, where $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ are arrows of \mathcal{C} , is a square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g_1 \downarrow & & \downarrow g_2 \\ A' & \xrightarrow{f'} & B' \end{array}$$

where $g_1 : A \rightarrow A'$ and $g_2 : B \rightarrow B'$ are arrows in \mathcal{C} , which is commutative in the sense that

$$f' \circ g_1 = g_2 \circ f$$

- composition: for every composable arrows (h_1, h_2) and (g_1, g_2) in $\mathcal{C}^{\rightarrow}$, their composite is defined as

$$(h_1, h_2) \circ (g_1, g_2) = (h_1 \circ g_1, h_2 \circ g_2).$$

- identity arrow: for every object $f : A \rightarrow B$ in $\mathcal{C}^{\rightarrow}$, the identity arrow is $1_f = (1_A, 1_B)$.

1.6 Free Categories

Let A be a set, which will be called an “alphabet”. Denote by A^* the set of all “words” with “letters” from A , that is, strings of elements from A . We call A^* the **Kleene closure** of A . Denote by e the empty word. We immediately have the following result.

Theorem 1.5. Let A be a set. Consider on A^* the operation “.” defined by concatenation. Then (A^*, \cdot) is a monoid with identity element e , called the free monoid on A .

Theorem 1.6 (Universal Mapping Property of the Free Monoid). With the above notation, there is an injective monoid homomorphism $i : A \hookrightarrow A^*$ with the property that for every monoid N and for every function $f : A \rightarrow N$, there is a unique monoid homomorphism $\bar{f} : A^* \rightarrow N$ such that $\bar{f} \circ i = f$, that is, the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{i} & A^* \\ f \downarrow & \swarrow \bar{f} & \\ N & & \end{array}$$

Proof. Let $i : A \hookrightarrow A^*$ be the inclusion homomorphism, which is an injective monoid homomorphism. Define $\bar{f} : A^* \rightarrow N$ by

$$\begin{aligned} \bar{f}(e) &= e_N \\ \bar{f}(w) &= f(a_1) \cdots f(a_n), \quad \forall w = a_1 \cdots a_n \in A^* \end{aligned}$$

One checks that \bar{f} is a monoid homomorphism and $\bar{f}(a) = f(a)$ for every $a \in A$, that is, $\bar{f} \circ i = f$.

For uniqueness, suppose that there is another monoid homomorphism $g : A^* \rightarrow N$ such that $g \circ i = f$. For every $w = a_1 \cdots a_n \in A^*$ we have

$$g(w) = g(a_1 \cdots a_n) = g(a_1) \cdots g(a_n) = f(a_1) \cdots f(a_n) = \bar{f}(a_1 \cdots a_n),$$

hence $\bar{f} = g$. □

Corollary 1.7. Universal mapping property of the free monoid determines it uniquely up to an isomorphism.

Proof. Suppose that M and N are free monoids on a set A . Consider the inclusion homomorphisms $i : A \rightarrow M$ and $j : A \rightarrow N$. Since M is a free monoid, by universal mapping property there is a monoid homomorphism $\alpha : M \rightarrow N$ such that $\alpha \circ i = j$. Since N is a free monoid, by universal mapping property there is a monoid homomorphism $\beta : N \rightarrow M$ such that $\beta \circ j = i$. It follows that $(\beta \circ \alpha) \circ i = i$. But we also have $1_M \circ i = i$. Then by the uniqueness from universal mapping property we must have $\beta \circ \alpha = 1_M$.

$$\begin{array}{ccc}
 A \xrightarrow{i} M & A \xrightarrow{j} N & A \xrightarrow{i} M \\
 j \downarrow \swarrow \alpha & i \downarrow \swarrow \beta & i \downarrow \swarrow \beta \circ \alpha = 1_M \\
 N & M & M
 \end{array}$$

Similarly, we have $\alpha \circ \beta = 1_N$. Hence $\alpha : M \rightarrow N$ is a monoid isomorphism. □

Next let us see how can we generalize the above results to categories.

To each category \mathcal{C} we may associate a graph $G = (V, E)$, where the class V of vertices consists of the objects of \mathcal{C} , while the class E of edges consists of the arrows of \mathcal{C} . Then we have two functions $s : E \rightarrow V$ (source) defined by $s(e) = v_1$ for every arrow $e : v_1 \rightarrow v_2$, and $t : E \rightarrow V$ (target) defined by $t(e) = v_2$ for every arrow $e : v_1 \rightarrow v_2$.

We define the free category on the graph G , denoted by $\mathcal{C}(G)$, as follows:

- objects: the vertices of G
- arrows: the paths in G
- composition: the concatenation of paths in G
- identity arrow: for every $v \in V$ the identity arrow 1_v is the loop on v .

We may define a functor $U : \mathbf{Cat} \rightarrow \mathbf{Graph}$, called the **forgetful functor**, which associates to a category \mathcal{C} its underlying graph having as edges the arrows of \mathcal{C} , and as vertices the objects of \mathcal{C} , and to a functor between categories its underlying graph homomorphism (that is, a functor without the conditions on composition and identity). One may prove the following result.

Theorem 1.8 (Universal Mapping Property of the Free Category on a Graph). With the above notation, there is a graph homomorphism $i : G \rightarrow U(\mathcal{C}(G))$ with the property that for every category \mathcal{D} and for every graph homomorphism $f : G \rightarrow U(\mathcal{D})$, there is a unique functor $\bar{f} : \mathcal{C}(G) \rightarrow \mathcal{D}$ such that $U(\bar{f}) \circ i = f$, that is, the following diagram is commutative:

$$\begin{array}{ccc}
 G & \xrightarrow{i} & U(\mathcal{C}(G)) \\
 f \downarrow & \swarrow U(\bar{f}) & \\
 U(\mathcal{D}) & &
 \end{array}$$

Corollary 1.9. Universal mapping property of the free category on a graph determines it uniquely up to a graph isomorphism.

1.7 Large, Small and Locally Small Categories

Definition 1.4. A category is called:

- (1) **small** if both classes of objects and arrows are sets.
- (2) **large** if it is not small.
- (3) **locally small** if for every objects $C, D \in \mathcal{C}$, $\text{Hom}_{\mathcal{C}}(C, D)$ is a set.

Example 1.1.

- (1) The category of finite sets and functions is equivalent to a small category.
- (2) **Set**, **Pos**, **Grp** and **Top** are locally small, but not small.

- (3) **Cat** is large, but not locally small. Indeed, if \mathcal{C} is a locally small category which is not small, and $\mathbf{1}$ is the category with one object and one arrow, then functors $\mathbf{1} \rightarrow \mathcal{C}$ are simply objects of \mathcal{C} , so $\text{Hom}_{\mathbf{Cat}}(\mathbf{1}, \mathcal{C})$ is not a set.

Chapter 2

Abstract Structure

2.1 Epimorphisms and Monomorphisms

Definition 2.1. Let \mathcal{C} be a category. An arrow $f : A \rightarrow B$ in \mathcal{C} is called a:

- (1) **monomorphism** (or briefly *mono*) if for every arrows $g, h : C \rightarrow A$ such that $f \circ g = f \circ h$, we have $g = h$.
- (2) **epimorphism** (or briefly *epi*) if for every arrows $g, h : B \rightarrow C$ such that $g \circ f = h \circ f$, we have $g = h$.
- (3) **bimorphism** if it is both a monomorphism and an epimorphism.

Lemma 2.1. Every isomorphism is a bimorphism.

Proof. Let $f : A \rightarrow B$ be an isomorphism in a category \mathcal{C} . Let $g, h : C \rightarrow A$ be arrows in \mathcal{C} such that $f \circ g = f \circ h$. Compose on the left by f^{-1} in order to get $g = h$. Hence f is a monomorphism. Similarly, one shows that f is an epimorphism. Thus, f is a bimorphism. \square

One may also prove the following property.

Proposition 2.2. Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be arrows in a category \mathcal{C} .

- (1) If f, g are monomorphisms (epimorphisms), then so is $g \circ f$.
- (2) If $g \circ f$ is monomorphism, then so is f .
- (3) If $g \circ f$ is epimorphism, then so is g .

Example 2.1.

- (1) In **Set** monomorphisms coincide with injective functions, epimorphisms coincide with surjective functions, and bimorphisms coincide with isomorphisms and with bijective functions.
- (2) In many usual concrete categories, monomorphisms coincide with injective arrows. In not so many usual categories epimorphisms coincide with surjective arrows.

For instance, in **Mon** monomorphisms coincide with injective monoid homomorphisms, but epimorphisms do not coincide with surjective monoid homomorphisms. Indeed, the inclusion map $i : \mathbb{N} \rightarrow \mathbb{Z}$, defined by $i(x) = x$, is an epimorphism in **Mon**, but it is not surjective. This is also an example of a bimorphism, which is not an isomorphism.

- (3) There is a large class of categories, called **abelian categories** (e.g., the categories **Ab** and $\text{Vect}(K)$), in which bimorphisms coincide with isomorphisms.

Definition 2.2. Let \mathcal{C} be a category. An arrow $f : A \rightarrow B$ in \mathcal{C} is called a:

- (1) **section** (or **split monomorphism**) if it has a left inverse arrow, that is, there is an arrow $g : B \rightarrow A$ such that $g \circ f = 1_A$.

- (2) **retraction** (or **split epimorphism**) if it has a right inverse arrow, that is, there is an arrow $g : B \rightarrow A$ such that $f \circ g = 1_B$.

Remark 2.3. An arrow is an isomorphism if and only if it is a section and a retraction.

Lemma 2.4.

- (1) Every section is a monomorphism.
 (2) Every retraction is an epimorphism.

Proof.

- (1) Let $f : A \rightarrow B$ be a section in a category \mathcal{C} . Hence there is an arrow $r : B \rightarrow A$ such that $r \circ f = 1_A$. Let $g, h : B \rightarrow C$ be arrows in \mathcal{C} such that $f \circ g = f \circ h$. Then we have

$$r \circ f \circ g = r \circ f \circ h,$$

which implies that $g = h$. Hence f is a monomorphism.

- (2) Similarly, one shows that every retraction is an epimorphism. □

One may show the following property.

Proposition 2.5. The following are equivalent for an arrow $f : A \rightarrow B$ in any category \mathcal{C} :

- (1) f is an isomorphism.
 (2) f is both a monomorphism and a retraction.
 (3) f is both a section and an epimorphism.

Example 2.2.

- (1) In **Set** every monomorphism (i.e., injective function) is a section, except those of the form $\emptyset \rightarrow A$ with $A \neq \emptyset$. In **Set** the condition that every epimorphism is a retraction is equivalent to the axiom of choice.
 (2) In **Ab** the inclusion homomorphism $i : 2\mathbb{Z} \rightarrow \mathbb{Z}$, defined by $i(2x) = 2x$, is clearly a monomorphism. Suppose that it is a section. Then there is a group homomorphism $g : \mathbb{Z} \rightarrow 2\mathbb{Z}$ such that $g \circ i = 1_{2\mathbb{Z}}$. We have $2g(1) = g(2) = g(i(2)) = 2$, hence $1 = g(1) \in 2\mathbb{Z}$, a contradiction. Therefore, there are monomorphisms, which are not sections.

In **Ab** the homomorphism $f : \mathbb{Z}_4 \rightarrow \mathbb{Z}_2$ defined by $f(\bar{x}) = \bar{x}$ is clearly an epimorphism. Suppose that it is a retraction. Then there is a group homomorphism $g : \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ such that $f \circ g = 1_{\mathbb{Z}_2}$. The order of $g(\hat{1})$ divides the order of $\hat{1}$, which is 2. Hence $g(\hat{1}) \in \{\bar{0}, \bar{2}\}$. But then $f(g(\hat{1})) = \hat{0} \neq \hat{1} = 1_{\mathbb{Z}_2}(\hat{1})$, a contradiction. Therefore, there are epimorphisms which are not retractions.

Proposition 2.6. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a (covariant) functor. Then:

- (1) F preserves sections in the sense that if f is a section, then $F(f)$ is also a section.
 (2) F preserves retractions in the sense that if f is a retraction, then $F(f)$ is also a retraction.
 (3) F preserves isomorphisms in the sense that if f is an isomorphism, then $F(f)$ is also an isomorphism.

Proof. Let $f : A \rightarrow B$ be a section in \mathcal{C} . Then there is an arrow $g : B \rightarrow A$ such that $g \circ f = 1_A$. This implies that

$$F(g) \circ F(f) = F(g \circ f) = F(1_A) = 1_{F(A)}$$

hence $F(f)$ is a section in \mathcal{D} . Similarly, F preserves retractions. The fact that F preserves isomorphisms is a consequence of the first two properties. □

2.2 Initial and Terminal Objects

Definition 2.3. Let \mathcal{C} be a category. An object C' of \mathcal{C} is called:

- (1) **initial** if for every object C of \mathcal{C} , there is a unique arrow $C' \rightarrow C$.
- (2) **terminal** if for every object C of \mathcal{C} , there is a unique arrow $C \rightarrow C'$.

Sometimes an initial object is denoted by 0 , while a terminal object is denoted by 1 .

Proposition 2.7. Initial and terminal objects are unique up to an isomorphism.

Proof. Assume that C', C'' are initial objects of a category \mathcal{C} . Then there is a unique arrow $f : C' \rightarrow C''$ and a unique arrow $g : C'' \rightarrow C'$. Note that we have the arrows $g \circ f : C' \rightarrow C'$ and $1_{C'} : C' \rightarrow C'$. Since C' is an initial object, we must have $g \circ f = 1_{C'}$. Also, note that we have the arrows $f \circ g : C'' \rightarrow C''$ and $1_{C''} : C'' \rightarrow C''$. Since C'' is an initial object, we must have $f \circ g = 1_{C''}$. Hence $f : C' \rightarrow C''$ is an isomorphism. Hence initial objects are unique up to an isomorphism.

Similarly, one shows that terminal objects are unique up to an isomorphism. \square

Example 2.3.

- (1) In **Set** the initial object is \emptyset , while the terminal object is any single-element set.
- (2) In **Grp** the initial object is the trivial group, while the terminal object is again the trivial group.
- (3) In **Ring** the initial object is \mathbb{Z} , while the terminal object is the trivial ring. Note that there is a unique unitary ring homomorphism $\mathbb{Z} \rightarrow R$ for every ring R with identity $1'$, which is defined by

$$f(n) = n \cdot 1', \quad \forall x \in \mathbb{Z}$$

We first show that if f does exist, then it is unique. So, suppose that $f : \mathbb{Z} \rightarrow R$ is a unitary ring homomorphism. Then $f(0) = 0' = 0 \cdot 1'$, where $0'$ is the zero element of R . For every $k \in \mathbb{N}^*$, we have:

$$\begin{aligned} f(k) &= f(\underbrace{1 + \cdots + 1}_{k \text{ times}}) = \underbrace{f(1) + \cdots + f(1)}_{k \text{ times}} = \underbrace{1' + \cdots + 1'}_{k \text{ times}} = k \cdot 1' \\ f(-k) &= -f(k) = -(k \cdot 1') = (-k) \cdot 1' \end{aligned}$$

Hence $f(n) = n \cdot 1'$ for every $n \in \mathbb{Z}$.

Now we show that the function f is a unitary ring homomorphism. For every $m, n \in \mathbb{Z}$, we have:

$$\begin{aligned} f(m+n) &= (m+n) \cdot 1' = m \cdot 1' + n \cdot 1' = f(m) + f(n), \\ f(m \cdot n) &= (m \cdot n) \cdot 1' = (m \cdot 1') \cdot (n \cdot 1') = f(m) \cdot f(n) \end{aligned}$$

and $f(1) = 1 \cdot 1' = 1'$. Hence f is a unitary ring homomorphism.

- (4) View the poset (\mathbb{Z}, \leq) as a poset category. This category has neither an initial object, nor a terminal object.

2.3 Products

Definition 2.4. Let \mathcal{C} be a category and A, B objects of \mathcal{C} . A **product diagram** for A and B consists of an object P and arrows, called **canonical projections**

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B$$

satisfying the following universal mapping property: given any diagram of the form

$$A \xleftarrow{f_1} X \xrightarrow{f_2} B$$

there is a unique arrow $u : X \rightarrow P$ such that

$$p_1 \circ u = f_1 \quad \text{and} \quad p_2 \circ u = f_2$$

that is, the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ f_1 \swarrow & \downarrow u & \searrow f_2 \\ A & \xleftarrow{p_1} P \xrightarrow{p_2} & B \end{array}$$

We denote the product of A and B by (P, p_1, p_2) . Sometimes P is also denoted by $A \amalg B$ or $A \times B$.

Remark 2.8. Sometimes (especially in the so-called additive categories), the canonical projections $p_1 : A \times B \rightarrow A$ and $p_2 : A \times B \rightarrow B$ are also denoted by $[1 \ 0] : A \times B \rightarrow A$ and $[0 \ 1] : A \times B \rightarrow B$ respectively. The unique arrow $u : X \rightarrow A \times B$ such that $p_1 \circ u = f_1$ and $p_2 \circ u = f_2$ is also denoted by

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : X \rightarrow A \times B$$

Then equalities involving compositions of arrows such as $p_1 \circ u = f_1$ and $p_2 \circ u = f_2$ may be rewritten in terms of matrix multiplications as $[1 \ 0] \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = f_1$ and $[0 \ 1] \cdot \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = f_2$.

Remark 2.9. Note that the canonical projections p_1 and p_2 need not be epimorphisms. For instance, consider the category described by the following graph:

$$A \xleftarrow{p_1} P \xrightarrow{p_2} B \xrightarrow[h]{g} C$$

such that $g \circ p_2 = h \circ p_2$. Then (P, p_1, p_2) is a product of A and B , but p_2 is not an epimorphism, because we have $g \circ p_2 = h \circ p_2$ and $g \neq h$.

As usual, universal mapping property insures the following uniqueness result.

Proposition 2.10. The product is unique up to an isomorphism.

Proof. Suppose that (P, p_1, p_2) and (Q, q_1, q_2) are products of objects A and B of a category \mathcal{C} . Since (Q, q_1, q_2) is a product, there is a unique arrow $i : P \rightarrow Q$ such that $q_1 \circ i = p_1$ and $q_2 \circ i = p_2$. Since (P, p_1, p_2) is a product, there is a unique arrow $j : Q \rightarrow P$ such that $p_1 \circ j = q_1$ and $p_2 \circ j = q_2$. Hence we have the following commutative diagram:

$$\begin{array}{ccccc} & P & & & \\ p_1 \swarrow & \downarrow i & \searrow p_2 & & \\ A & \xleftarrow{q_1} Q \xrightarrow{q_2} & B & & \\ p_1 \swarrow & \downarrow j & \searrow p_2 & & \\ & P & & & \end{array}$$

It follows that $p_1 \circ (j \circ i) = p_1$ and $p_2 \circ (j \circ i) = p_2$. But we also have $p_1 \circ 1_P = p_1$ and $p_2 \circ 1_P = p_2$, and by the uniqueness condition of universal mapping property we have $j \circ i = 1_P$. Similarly, one shows that $i \circ j = 1_Q$. Hence $i : P \rightarrow Q$ is an isomorphism. \square

More generally, one may define a product of an arbitrary family of objects of a category, which is again unique up to an isomorphism.

Definition 2.5. A **product** of a family $(A_i)_{i \in I}$ of objects of a category \mathcal{C} consists of an object P , also denoted by $\prod_{i \in I} A_i$, and a family $(p_i)_{i \in I}$ of arrows, where $p_i : P \rightarrow A_i$ for every $i \in I$, satisfying

the following universal mapping property: given any object X of \mathcal{C} and any family $(f_i)_{i \in I}$ of arrows, where $f_i : X \rightarrow A_i$ for every $i \in I$, there is a unique arrow $u : X \rightarrow P$ such that $p_i \circ u = f_i$ for every $i \in I$.

2.4 Examples of Products

2.4.1 The Category Set

The product of two sets A and B is

$$(A \times B, p_1, p_2)$$

where $p_1 : A \times B \rightarrow A$ is the function defined by $p_1(a, b) = a$, and $p_2 : A \times B \rightarrow B$ is the function defined by $p_2(a, b) = b$.

Let X be a set and let $f_1 : X \rightarrow A$ and $f_2 : X \rightarrow B$ be functions. We look for a unique function $u : X \rightarrow A \times B$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ f_1 \swarrow & \vdots u & \searrow f_2 \\ A & \xleftarrow{p_1} & A \times B \xrightarrow{p_2} B \end{array}$$

that is, $p_1 \circ u = f_1$ and $p_2 \circ u = f_2$. These equalities are equivalent to $p_1(u(x)) = f_1(x)$ and $p_2(u(x)) = f_2(x)$ for every $x \in X$. This means that

$$u(x) = (f_1(x), f_2(x)), \quad \forall x \in X$$

Note that u is uniquely determined by this definition.

The construction of a product may be easily generalized to an arbitrary family of sets.

2.4.2 The Category Grp

The product of two groups (G_1, \cdot) and (G_2, \cdot) is

$$((G_1 \times G_2, \cdot), p_1, p_2)$$

where $p_1 : G_1 \times G_2 \rightarrow G_1$ is the function defined by $p_1(g_1, g_2) = g_1$, and $p_2 : G_1 \times G_2 \rightarrow G_2$ is the function defined by $p_2(g_1, g_2) = g_2$.

Note that $G_1 \times G_2$ is a group with respect to the operation defined by

$$(x_1, x_2) \cdot (x'_1, x'_2) = (x_1 \cdot x'_1, x_2 \cdot x'_2), \quad \forall (x_1, x_2), (x'_1, x'_2) \in G_1 \times G_2$$

Let (X, \cdot) be a group and let $f_1 : X \rightarrow G_1$ and $f_2 : X \rightarrow G_2$ be group homomorphisms. We look for a unique group homomorphism $u : X \rightarrow G_1 \times G_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} & X & \\ f_1 \swarrow & \vdots u & \searrow f_2 \\ G_1 & \xleftarrow{p_1} & G_1 \times G_2 \xrightarrow{p_2} G_2 \end{array}$$

that is, $p_1 \circ u = f_1$ and $p_2 \circ u = f_2$. These equalities are equivalent to $p_1(u(x)) = f_1(x)$ and $p_2(u(x)) = f_2(x)$ for every $x \in X$. This means that

$$u(x) = (f_1(x), f_2(x)), \quad \forall x \in X$$

Note that u is uniquely determined by this definition.

We still need to prove that u is a group homomorphism. For every $x_1, x_2 \in X$ we have

$$\begin{aligned} u(x_1 \cdot x_2) &= (f_1(x_1 \cdot x_2), f_2(x_1 \cdot x_2)) = (f_1(x_1) \cdot f_1(x_2), f_2(x_1) \cdot f_2(x_2)) \\ &= (f_1(x_1), f_2(x_1)) \cdot (f_1(x_2), f_2(x_2)) = u(x_1) \cdot u(x_2) \end{aligned}$$

hence u is a group homomorphism.

The construction of a product may be easily generalized to an arbitrary family of groups.

2.4.3 Poset Categories

Let (L, \leq) be a lattice. Hence every two elements of L have an infimum (i.e., greatest lower bound). Since (L, \leq) is a poset, we may view it as a poset category. Recall that its objects are the elements of L , while an arrow $x \rightarrow y$ does exist if and only if $x \leq y$, where $x, y \in L$.

The product of two elements $x, y \in L$ is

$$(\inf(x, y), p_1, p_2)$$

where $p_1 : \inf(x, y) \rightarrow x$ and $p_2 : \inf(x, y) \rightarrow y$ are the unique arrows having the given domains and codomains.

Let $z \in L$ and let $z \rightarrow x$ and $z \rightarrow y$ be arrows. This means that $z \leq x$ and $z \leq y$. We look for a unique arrow $u : z \rightarrow \inf(x, y)$ such that the following diagram is commutative:

$$\begin{array}{ccc} & z & \\ & \swarrow & \searrow \\ x & \longleftarrow \inf(x, y) \longrightarrow & y \end{array}$$

This means that $z \leq \inf(x, y)$. But this is true, because z is a lower bound of x and y , while $\inf(x, y)$ is the greatest lower bound of x and y .

Note that if a poset (L, \leq) is not a lattice, two elements of L might not have a product.

The construction of a product may be easily generalized to an arbitrary family of elements, when (L, \leq) is a complete lattice, that is, every family of elements of L has an infimum and a supremum.

2.5 Categories with Products

Definition 2.6. A category \mathcal{C} is said to **have (binary) products** if any family of (two) objects of \mathcal{C} has a product.

Example 2.4. We have seen that **Set** and **Grp** have (binary) products, while poset categories may not have products.

Let \mathcal{C} be a category with binary products, and let $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ be arrows in \mathcal{C} . We want to define a product $f \times f' : A \times A' \rightarrow B \times B'$ of f and f' .

Consider the products $(A \times A', p_1, p_2)$ and $(B \times B', q_1, q_2)$. Let $f_1 = f \circ p_1$ and $f_2 = f' \circ p_2$. By universal mapping property of the product $(B \times B', q_1, q_2)$ there is a unique arrow $u : A \times A' \rightarrow B \times B'$ such that the following diagram is commutative:

$$\begin{array}{ccccc} A & \xleftarrow{p_1} & A \times A' & \xrightarrow{p_2} & A' \\ f \downarrow & & \downarrow u & & \downarrow f' \\ B & \xleftarrow{q_1} & B \times B' & \xrightarrow{q_2} & B' \end{array}$$

that is, $q_1 \circ u = f_1 = f \circ p_1$ and $q_2 \circ u = f_2 = f' \circ p_2$. We define $f \times f' = u : A \times A' \rightarrow B \times B'$.

One may prove the following result.

Proposition 2.11. Let \mathcal{C} be a category with binary products. Then we have a covariant functor $R : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$R(C, C') = C \times C'$$

for every object (C, C') of $\mathcal{C} \times \mathcal{C}$, and

$$R(f, f') = f \times f' : A \times A' \rightarrow B \times B'$$

for every arrow (f, f') from $\mathcal{C} \times \mathcal{C}$ with $f : A \rightarrow B$ and $f' : A' \rightarrow B'$.

For a category \mathcal{C} with products, one may generalize this construction to any finite family of arrows, and define a corresponding functor.

One may show the following associativity property by using universal mapping property of the product.

Proposition 2.12. In any category \mathcal{C} with binary products, we have

$$A \times (B \times C) \cong (A \times B) \times C,$$

where A, B, C are objects of \mathcal{C} .

2.6 Hom-sets

In this section, assume that all categories are locally small, that is, $\text{Hom}_{\mathcal{C}}(A, B)$ is a set for every $A, B \in \mathcal{C}$.

Let A be an object and $f : B \rightarrow B'$ an arrow in a category \mathcal{C} . We define

$$\begin{aligned} f_* &= \text{Hom}_{\mathcal{C}}(A, f) : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, B') \\ f_*(g) &= f \circ g, \quad \forall g \in \text{Hom}_{\mathcal{C}}(A, B) \end{aligned}$$

One may show the following property.

Proposition 2.13. Let \mathcal{C} be a category and let A be an object of \mathcal{C} . Then we have a covariant functor $\text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$, called the covariant representable functor, defined by

$$B \mapsto \text{Hom}_{\mathcal{C}}(A, B)$$

on every object B of \mathcal{C} and

$$f \mapsto f_* = \text{Hom}_{\mathcal{C}}(A, f) : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{C}}(A, B')$$

for every arrow $f : B \rightarrow B'$ in \mathcal{C} .

One may show the following property.

Proposition 2.14. Let \mathcal{C} be a category with binary products. Then for every object A of \mathcal{C} , the covariant functor $\text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$ preserves binary products, that is, for every $C, D \in \mathcal{C}$, there is a bijection (i.e., isomorphism in \mathbf{Set}):

$$\text{Hom}_{\mathcal{C}}(A, C \times D) \cong \text{Hom}_{\mathcal{C}}(A, C) \times \text{Hom}_{\mathcal{C}}(A, D).$$

Definition 2.7. A contravariant functor between two categories \mathcal{C} and \mathcal{D} is a mapping of objects of \mathcal{C} to objects of \mathcal{D} and of arrows of \mathcal{C} to arrows of \mathcal{D} , denoted by

$$F : \mathcal{C} \rightarrow \mathcal{D}$$

satisfying the axioms:

- (i) For every $f : A \rightarrow B$ in \mathcal{C} , we have $F(f) : F(B) \rightarrow F(A)$ in \mathcal{D} .
- (ii) For every object A of \mathcal{C} , we have $F(1_A) = 1_{F(A)}$.
- (iii) For every composable pair of arrows $f : A \rightarrow B$ and $g : B \rightarrow C$ in \mathcal{C} , we have

$$F(g \circ f) = F(f) \circ F(g)$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{g \circ f} & \downarrow g \\ & & C \end{array} \quad \Rightarrow \quad \begin{array}{ccc} F(A) & \xleftarrow{F(f)} & F(B) \\ & \swarrow^{F(g \circ f)} & \uparrow^{F(g)} \\ & & F(C) \end{array}$$

Let A be an object and $f : B \rightarrow B'$ an arrow in a category \mathcal{C} . We define

$$\begin{aligned} f^* &= \text{Hom}_{\mathcal{C}}(f, A) : \text{Hom}_{\mathcal{C}}(B', A) \rightarrow \text{Hom}_{\mathcal{C}}(B, A), \\ f^*(g) &= g \circ f, \quad \forall g \in \text{Hom}_{\mathcal{C}}(B', A). \end{aligned}$$

Proposition 2.15. Let \mathcal{C} be a category and let A be an object of \mathcal{C} . Then we have a contravariant functor $\text{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \rightarrow \mathbf{Set}$, called the **contravariant representable functor**, defined by

$$B \mapsto \text{Hom}_{\mathcal{C}}(B, A)$$

on every object B of \mathcal{C} and

$$f \mapsto f^* = \text{Hom}_{\mathcal{C}}(f, A) : \text{Hom}_{\mathcal{C}}(B', A) \rightarrow \text{Hom}_{\mathcal{C}}(B, A)$$

on every arrow $f : B \rightarrow B'$ in \mathcal{C} .

Proof.

(i) For every arrow $f : B \rightarrow B'$ in \mathcal{C} , we have the following function (arrow in \mathbf{Set}):

$$f^* = \text{Hom}_{\mathcal{C}}(f, A) : \text{Hom}_{\mathcal{C}}(B', A) \rightarrow \text{Hom}_{\mathcal{C}}(B, A)$$

(ii) For every object B of \mathcal{C} , we have

$$\text{Hom}_{\mathcal{C}}(1_B, A) = 1_B^* = 1_{\text{Hom}_{\mathcal{C}}(B, A)}^{\mathcal{C}}$$

(iii) Let $f : B \rightarrow B'$ and $g : B' \rightarrow B''$ be arrows in \mathcal{C} . Then we have the following functions (arrows in \mathbf{Set})

$$(g \circ f)^*, f^* \circ g^* : \text{Hom}_{\mathcal{C}}(B'', A) \rightarrow \text{Hom}_{\mathcal{C}}(B, A)$$

For every function $h \in \text{Hom}_{\mathcal{C}}(B'', A)$, we have

$$(g \circ f)^*(h) = h \circ (g \circ f) = (h \circ g) \circ f = g^*(h) \circ f = f^*(g^*(h)) = (f^* \circ g^*)(h)$$

Hence we have $(g \circ f)^* = f^* \circ g^*$. It follows that

$$\text{Hom}_{\mathcal{C}}(g \circ f, A) = (g \circ f)^* = f^* \circ g^* = \text{Hom}_{\mathcal{C}}(f, A) \circ \text{Hom}_{\mathcal{C}}(g, A)$$

Hence $\text{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \rightarrow \mathbf{Set}$ is a contravariant functor. □

Chapter 3

Duality

3.1 The Duality Principle

In the formal definition of a category there are objects A, B, C, \dots , arrows f, g, h, \dots and four operations given by $\text{dom}(f), \text{cod}(f), 1_A, g \circ f$ which satisfy the following seven axioms:

- $\text{dom}(1_A) = A$.
- $\text{cod}(1_A) = A$.
- $f \circ 1_{\text{dom}(f)} = f$.
- $1_{\text{cod}(f)} \circ f = f$.
- $\text{dom}(g \circ f) = \text{dom}(f)$.
- $\text{cod}(g \circ f) = \text{cod}(g)$.
- $h \circ (g \circ f) = (h \circ g) \circ f$.

Of course, the operation “ $g \circ f$ ” is only defined when $\text{dom}(g) = \text{cod}(f)$.

Given any sentence Σ in the elementary language of category theory, we can form the “dual statement” Σ^* by making the following replacements: $f \circ g$ for $g \circ f$, cod for dom , dom for cod . It is easy to see that then Σ^* will also be a well-formed sentence. Next, suppose we have shown a sentence Σ to entail one Δ , that is,

$$\Sigma \implies \Delta,$$

without using any of the category axioms, then clearly

$$\Sigma^* \implies \Delta^*$$

since the substituted terms are treated as mere undefined constants. But now observe that the axioms for category theory (CT) are themselves “self-dual” in the sense that we have $CT^* = CT$.

Therefore we have the following formal duality principle.

Proposition 3.1 (Formal duality). For any sentence Σ in the language of category theory (CT) if Σ follows from the axioms of categories, then its dual Σ^* also follows, i.e.,

$$(CT \implies \Sigma) \text{ implies } (CT \implies \Sigma^*)$$

Now assume that Σ holds for any category \mathcal{C} . Then Σ holds for any opposite category \mathcal{C}^{op} . Hence Σ^* holds in $\mathcal{C} = (\mathcal{C}^{\text{op}})^{\text{op}}$ for any category \mathcal{C} .

Therefore we have the following conceptual form of the duality principle.

Proposition 3.2 (Conceptual duality). For any statement Σ about categories, if Σ holds for all categories, then Σ^* holds for all categories.

3.2 Coproducts

Definition 3.1. Let \mathcal{C} be a category and let A and B be objects of \mathcal{C} . A coproduct of A and B in \mathcal{C} is just the product of A and B in the opposite category \mathcal{C}^{op} . This means an object Q and arrows $q_1 : A \rightarrow Q$ and $q_2 : B \rightarrow Q$, called canonical injections, satisfying the following universal mapping property: given any diagram of the form

$$A \xrightarrow{f_1} Z \xleftarrow{f_2} B$$

there is a unique arrow $u : Q \rightarrow Z$ such that

$$u \circ q_1 = f_1 \quad \text{and} \quad u \circ q_2 = f_2$$

that is, the following diagram is commutative:

$$\begin{array}{ccc} & Z & \\ f_1 \nearrow & \uparrow u & \nwarrow f_2 \\ A & \xrightarrow{q_1} Q \xleftarrow{q_2} & B \end{array}$$

We denote the coproduct of A and B by (Q, q_1, q_2) . Sometimes Q is also denoted by $A \coprod B$ or $A \oplus B$.

Remark 3.3. Sometimes (especially in the so-called additive categories), similarly to the case of products, the canonical injections $q_1 : A \rightarrow A \oplus B$ and $q_2 : B \rightarrow A \oplus B$ are also denoted by $\begin{bmatrix} 1 \\ 0 \end{bmatrix} : A \rightarrow A \oplus B$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} : B \rightarrow A \oplus B$ respectively. The unique arrow $u : A \oplus B \rightarrow Z$ such that $u \circ q_1 = f_1$ and $u \circ q_2 = f_2$ is also denoted by

$$[f_1 \quad f_2] : A \oplus B \rightarrow Z$$

Then equalities involving compositions of arrows such as $u \circ q_1 = f_1$ and $u \circ q_2 = f_2$ may be rewritten in terms of matrix multiplications as $[f_1 \quad f_2] \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = f_1$ and $[f_1 \quad f_2] \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = f_2$.

Remark 3.4. Note that the canonical injections q_1 and q_2 need not be monomorphisms. Just consider the example with the canonical projections from products in the opposite category.

As usual, universal mapping property insures the following uniqueness result. It follows by the duality principle.

Proposition 3.5. The coproduct is unique up to an isomorphism.

More generally, one may define a coproduct of an arbitrary family of objects of a category, which is again unique up to an isomorphism.

Definition 3.2. A coproduct of a family $(A_i)_{i \in I}$ of objects of a category \mathcal{C} consists of an object Q , also denoted by $\coprod_{i \in I} A_i$ or $\bigoplus_{i \in I} A_i$, and a family $(q_i)_{i \in I}$ of arrows, where $q_i : A_i \rightarrow Q$ for every $i \in I$, satisfying the following universal mapping property: given any object Z of \mathcal{C} and any family $(f_i)_{i \in I}$ of arrows, where $f_i : A_i \rightarrow Z$ for every $i \in I$, there is a unique arrow $u : Q \rightarrow Z$ such that $u \circ q_i = f_i$ for every $i \in I$.

Next we present some examples of coproducts in certain categories.

3.2.1 The Category Set

The coproduct of two sets A and B is $(A \sqcup B, q_1, q_2)$, where

$$A \sqcup B = \{(a, 1) \mid a \in A\} \cup \{(b, 2) \mid b \in B\}$$

is the disjoint union of A and B , $q_1 : A \rightarrow A \sqcup B$ is the function defined by $q_1(a) = (a, 1)$, and $q_2 : B \rightarrow A \sqcup B$ is the function defined by $q_2(b) = (b, 2)$.

Let Z be a set and let $f_1 : A \rightarrow Z$ and $f_2 : B \rightarrow Z$ be functions. We look for a unique function $u : A \sqcup B \rightarrow Z$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
 & & Z & & \\
 & f_1 \nearrow & \uparrow u & \nwarrow f_2 & \\
 A & \xrightarrow{q_1} & A \sqcup B & \xleftarrow{q_2} & B
 \end{array}$$

that is, $u \circ q_1 = f_1$ and $u \circ q_2 = f_2$. These equalities are equivalent to $u(q_1(a)) = f_1(a)$ and $u(q_2(b)) = f_2(b)$ for every $a \in A$ and $b \in B$, and furthermore, $u(a, 1) = f_1(a)$ and $u(b, 2) = f_2(b)$ for every $a \in A$ and $b \in B$. Note that u is uniquely determined by this definition.

The construction of a coproduct may be easily generalized to an arbitrary family of sets.

3.2.2 The Category Ab

The coproduct of two abelian groups $(A, +)$ and $(B, +)$ is $((A \times B, +), q_1, q_2)$, where $q_1 : A \rightarrow A \times B$ is the group homomorphism defined by $q_1(a) = (a, 0)$, and $q_2 : B \rightarrow A \times B$ is the group homomorphism defined by $q_2(b) = (0, b)$.

Note that $A \times B$ is a group with respect to the operation defined by

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2), \quad \forall (a_1, b_1), (a_2, b_2) \in A \times B.$$

Let $(Z, +)$ be an abelian group and let $f_1 : A \rightarrow Z$ and $f_2 : B \rightarrow Z$ be group homomorphisms. We look for a unique group homomorphism $u : A \times B \rightarrow Z$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
 & & Z & & \\
 & f_1 \nearrow & \uparrow u & \nwarrow f_2 & \\
 A & \xrightarrow{q_1} & A \times B & \xleftarrow{q_2} & B
 \end{array}$$

that is, $u \circ q_1 = f_1$ and $u \circ q_2 = f_2$. These equalities are equivalent to $u(q_1(a)) = f_1(a)$ and $u(q_2(b)) = f_2(b)$ for every $a \in A$ and $b \in B$, and furthermore, $u(a, 0) = f_1(a)$ and $u(0, b) = f_2(b)$ for every $a \in A$ and $b \in B$. Note that a group homomorphism u is uniquely determined by this definition, because we have

$$u(a, b) = u((a, 0) + (0, b)) = u(a, 0) + u(0, b) = f_1(a) + f_2(b)$$

for every $a \in A$ and $b \in B$. One checks that the map u defined as above is really a group homomorphism.

The construction of a coproduct may be generalized to an arbitrary family of abelian groups, but in a slightly different manner. For a family $(A_i)_{i \in I}$ of abelian groups, we denote

$$\bigoplus_{i \in I} A_i = \left\{ (a_i)_{i \in I} \in \prod_{i \in I} A_i \mid (a_i)_{i \in I} \text{ has a finite number of non-zero elements} \right\}.$$

Note that if I is a finite set, then we have $\bigoplus_{i \in I} A_i \cong \prod_{i \in I} A_i$.

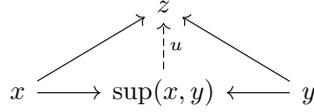
The coproduct of the family $(A_i)_{i \in I}$ of abelian groups is $\left(\bigoplus_{i \in I} A_i, (q_i)_{i \in I} \right)$, where for every $j \in I$, $q_j : A_j \rightarrow \bigoplus_{i \in I} A_i$ is the group homomorphism defined by $q_j(a) = (a_i)_{i \in I}$ with the properties that $a_j = a$ and $a_i = 0$ for every $i \in I$ with $i \neq j$.

3.2.3 Poset Categories

Let (L, \leq) be a lattice. Hence every two elements of L have a supremum (i.e., smallest upper bound). Since (L, \leq) is a poset, we may view it as a poset category. Recall that its objects are the elements of L , while an arrow $x \rightarrow y$ does exist if and only if $x \leq y$, where $x, y \in L$.

The coproduct of two elements $x, y \in L$ is $(\sup(x, y), q_1, q_2)$, where $q_1 : x \rightarrow \sup(x, y)$ and $q_2 : y \rightarrow \sup(x, y)$ are the unique arrows having the given domains and codomains.

Let $z \in L$ and let $x \rightarrow z$ and $y \rightarrow z$ be arrows. This means that $x \leq z$ and $y \leq z$. We look for a unique arrow $u : \sup(x, y) \rightarrow z$ such that the following diagram is commutative:



This means that $\sup(x, y) \leq z$. But this is true, because z is an upper bound of x and y , while $\sup(x, y)$ is the smallest upper bound of x and y .

Note that if a poset (L, \leq) is not a lattice, two elements of L might not have a coproduct.

The construction of a coproduct may be easily generalized to an arbitrary family of elements, when (L, \leq) is a complete lattice, that is, every family of elements of L has an infimum and a supremum.

3.2.4 Categories with Coproducts

Definition 3.3. A category \mathcal{C} is said to **have (binary) coproducts** if any family of (two) objects of \mathcal{C} has a coproduct.

Example 3.1. We have seen that **Set** and **Ab** have (binary) coproducts, while poset categories may not have coproducts.

Let \mathcal{C} be a category with binary coproducts, and let $f : A \rightarrow B$ and $f' : A' \rightarrow B'$ be arrows in \mathcal{C} . Dually to the case of products, one may define a coproduct $f \oplus f' : A \oplus A' \rightarrow B \oplus B'$ of f and f' .

Consider the coproducts $(A \oplus A', p_1, p_2)$ and $(B \oplus B', q_1, q_2)$. Let $f_1 = q_1 \circ f$ and $f_2 = q_2 \circ f'$. By universal mapping property of the product $(A \oplus A', p_1, p_2)$ there is a unique arrow $u : A \oplus A' \rightarrow B \oplus B'$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
 B & \xrightarrow{q_1} & B \oplus B' & \xleftarrow{q_2} & B' \\
 f \uparrow & & \uparrow u & & \uparrow f' \\
 A & \xrightarrow{p_1} & A \oplus A' & \xleftarrow{p_2} & A'
 \end{array}$$

that is, $u \circ p_1 = f_1 = q_1 \circ f$ and $u \circ p_2 = f_2 = q_2 \circ f'$. We define $f \oplus f' = u : A \oplus A' \rightarrow B \oplus B'$.

One may prove the following result.

Proposition 3.6. Let \mathcal{C} be a category with binary coproducts. Then we have a covariant functor $L : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ defined by

$$L(C, C') = C \oplus C'$$

for every object (C, C') of $\mathcal{C} \times \mathcal{C}$, and

$$L(f, f') = f \oplus f' : A \oplus A' \rightarrow B \oplus B'$$

for every arrow (f, f') from $\mathcal{C} \times \mathcal{C}$ with $f : A \rightarrow B$ and $f' : A' \rightarrow B'$.

For a category \mathcal{C} with coproducts, one may generalize this construction to any finite family of arrows, and define a corresponding functor.

One may show the following associativity property by using universal mapping property of the coproduct.

Proposition 3.7.

Proposition 3.2.9 In any category \mathcal{C} with binary coproducts, we have

$$A \oplus (B \oplus C) \cong (A \oplus B) \oplus C$$

where A, B, C are objects of \mathcal{C} .

One may show the following property.

Proposition 3.8. Let \mathcal{C} be a locally small category with binary coproducts. Then for every object A of \mathcal{C} , the contravariant functor $\text{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \rightarrow \mathbf{Set}$ preserves binary coproducts, that is, for every $B, C \in \mathcal{C}$, there is a bijection (i.e., isomorphism in **Set**):

$$\text{Hom}_{\mathcal{C}}(B \oplus C, A) \cong \text{Hom}_{\mathcal{C}}(B, A) \oplus \text{Hom}_{\mathcal{C}}(C, A)$$

3.3 Equalizers

Definition 3.4. Let \mathcal{C} be a category and let $f, g : A \rightarrow B$ be arrows in \mathcal{C} . An **equalizer** of f and g consists of a pair (E, e) , where E is an object of \mathcal{C} and $e : E \rightarrow A$ is an arrow in \mathcal{C} such that $f \circ e = g \circ e$ and it has the following universal mapping property: given any object Z and any arrow $z : Z \rightarrow A$ in \mathcal{C} such that $f \circ z = g \circ z$, there is a unique arrow $u : Z \rightarrow E$ in \mathcal{C} such that $e \circ u = z$.

$$\begin{array}{ccccc} E & \xrightarrow{e} & A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B \\ \uparrow \text{---} u \text{---} & \nearrow z & & & \\ Z & & & & \end{array}$$

By universal mapping property of an equalizer one deduces the following result.

Proposition 3.9. An equalizer is uniquely determined up to an isomorphism.

Proposition 3.10. Any equalizer is a monomorphism.

Proof. Let $f, g : A \rightarrow B$ be arrows in a category \mathcal{C} having an equalizer (E, e) . Let $\alpha, \beta : Z \rightarrow E$ be arrows in \mathcal{C} such that $e \circ \alpha = e \circ \beta$. Denote $z = e \circ \alpha = e \circ \beta : Z \rightarrow A$. Hence we have

$$f \circ z = f \circ e \circ \alpha = g \circ e \circ \alpha = g \circ e \circ \beta = g \circ z.$$

By universal mapping property of the equalizer, there is a unique arrow $u : Z \rightarrow E$ such that $e \circ u = z$. But we also have $e \circ \alpha = z$ and $e \circ \beta = z$. Hence we must have $u = \alpha = \beta$. This shows that e is a monomorphism. \square

3.3.1 The Category \mathbf{Set}

The equalizer of two functions $f, g : A \rightarrow B$ in \mathbf{Set} is the pair (E, e) , where

$$E = \{a \in A \mid f(a) = g(a)\}$$

and $e : E \rightarrow A$ is the inclusion function.

For every $a \in E$ we have

$$(f \circ e)(a) = f(e(a)) = f(a) = g(a) = g(e(a)) = (g \circ e)(a),$$

hence $f \circ e = g \circ e$.

Now let $z : Z \rightarrow A$ be a function such that $f \circ z = g \circ z$. We look for a function $u : Z \rightarrow E$ such that $e \circ u = z$. This equality is equivalent to $e(u(x)) = z(x)$ for every $x \in Z$, that is, $u(x) = z(x)$ for every $x \in Z$. Note that $z(x) \in E$, because $f(z(x)) = g(z(x))$. Also, the equality $u(x) = z(x)$ uniquely determines u . Hence (E, e) is an equalizer of f, g .

3.3.2 The Category $\mathbf{Vect}(K)$

The equalizer of two K -linear maps $f, g : A \rightarrow B$ in $\mathbf{Vect}(K)$ is the pair (E, e) , where

$$E = \{a \in A \mid f(a) = g(a)\}$$

and $e : E \rightarrow A$ is the inclusion K -linear map.

As in the category \mathbf{Set} , we have $f \circ e = g \circ e$. Also, for every K -linear map $z : Z \rightarrow A$ such that $f \circ z = g \circ z$, there is a unique function $u : Z \rightarrow E$ such that $e \circ u = z$. This equality is equivalent to $e(u(x)) = z(x)$ for every $x \in Z$, that is, $u(x) = z(x)$ for every $x \in Z$.

Let us show that u is a K -linear map. Let $k_1, k_2 \in K$ and $x_1, x_2 \in Z$. Then we have

$$u(k_1x_1 + k_2x_2) = z(k_1x_1 + k_2x_2) = k_1z(x_1) + k_2z(x_2) = k_1u(x_1) + k_2u(x_2)$$

Hence u is a K -linear map.

In this category the equalizer of two K -linear maps is in fact a kernel of some K -linear map, namely

$$E = \{a \in A \mid (f - g)(a) = 0\} = \text{Ker}(f - g)$$

On the other hand, the kernel of a K -linear map is the equalizer of some K -linear maps, namely

$$\text{Ker}(f) = \{a \in A \mid f(a) = 0\}$$

is the equalizer of the K -linear map $f : A \rightarrow B$ and the zero K -linear map $0 : A \rightarrow B$.

3.3.3 Monoid Categories

Let (M, \cdot) be a monoid. Recall that it may be viewed as a monoid category, where the single object is M , the arrows are the elements of M and the composition is the multiplication of the elements of M . An equalizer of two elements $a, b \in M$ is an element $x \in M$ with $ax = bx$ and for every $z \in M$ such that $az = bz$ there is $u \in M$ such that $xu = z$. If (M, \cdot) is a non-trivial group, then $a, b \in M$ with $a \neq b$ do not have an equalizer, because there is no $x \in M$ such that $ax = bx$.

Definition 3.5. We say that a category **has equalizers** if every arrows $f, g : A \rightarrow B$ have an equalizer.

Example 3.2. The categories **Set** and **Vect**(K) have equalizers, but monoid categories may not have equalizers.

3.4 Coequalizer

Definition 3.6. Let \mathcal{C} be a category and let $f, g : A \rightarrow B$ be arrows in \mathcal{C} . A coequalizer of f and g consists of a pair (Q, q) , where Q is an object of \mathcal{C} and $q : B \rightarrow Q$ is an arrow in \mathcal{C} such that $q \circ f = q \circ g$ and it has the following universal mapping property: given any object Z and any arrow $z : B \rightarrow Z$ in \mathcal{C} such that $z \circ f = z \circ g$, there is a unique arrow $u : Q \rightarrow Z$ in \mathcal{C} such that $u \circ q = z$.

$$\begin{array}{ccc} A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B & \xrightarrow{q} & Q \\ & & & \searrow z & \downarrow \text{---} u \\ & & & & Z \end{array}$$

The following two results are dual to the corresponding ones for equalizers.

Proposition 3.11. A coequalizer is uniquely determined up to an isomorphism.

Proposition 3.12. Any coequalizer is an epimorphism.

3.4.1 The Category Set

Let $f : A \rightarrow B$ be function. Then the homogeneous relation $\text{ker}(f) = (A, A, \text{Ker}(f))$ with the graph

$$\text{Ker}(f) = \{(x_1, x_2) \in A \times A \mid f(x_1) = f(x_2)\}$$

is called the **kernel** of f . Note that the kernel of f is an equivalence relation on A .

The following theorem will be useful.

Theorem 3.13 (Factorization theorem by a surjection). Let $f : A \rightarrow B$ be a function and let $g : A \rightarrow C$ be a surjective function such that $\text{Ker}(g) \subseteq \text{Ker}(f)$. Then there is a unique function $h : C \rightarrow B$ such that $f = h \circ g$.

Let $f, g : A \rightarrow B$ be functions. We define a relation $r = (B, B, R)$ by

$$(b_1, b_2) \in R \iff \exists a \in A : b_1 = f(a) \text{ and } b_2 = g(a).$$

Let $\bar{r} = (B, B, \bar{R})$ be the smallest equivalence relation on B containing R . Then we may consider the partition

$$B/\bar{R} = \{\bar{R}\langle b \rangle \mid b \in B\}$$

of B . Consider the function $\pi : B \rightarrow B/\bar{R}$ defined by $\pi(b) = \bar{R}\langle b \rangle$. Note that the kernel of the function π is

$$\begin{aligned} \text{Ker}(\pi) &= \{(b_1, b_2) \in B \times B \mid \pi(b_1) = \pi(b_2)\} \\ &= \{(b_1, b_2) \in B \times B \mid \bar{R}\langle b_1 \rangle = \bar{R}\langle b_2 \rangle\} = \bar{R}. \end{aligned}$$

We show that $(B/\bar{R}, \pi)$ is a coequalizer of f and g .

For every $a \in A$, we have $(f(a), g(a)) \in \bar{R}$, which implies that $\bar{R}\langle f(a) \rangle = \bar{R}\langle g(a) \rangle$. It follows that

$$(\pi \circ f)(a) = \bar{R}\langle f(a) \rangle = \bar{R}\langle g(a) \rangle = (\pi \circ g)(a),$$

hence $\pi \circ f = \pi \circ g$.

Now let $z : B \rightarrow Z$ be a function such that $z \circ f = z \circ g$. We show that

$$\bar{R} \subseteq \text{Ker}(z) = \{(b_1, b_2) \in B \times B \mid z(b_1) = z(b_2)\}$$

(the kernel of the function z). Let $(b_1, b_2) \in \bar{R}$. Then there is a finite number of elements $c_0, \dots, c_n \in B$ such that $b_1 = c_0, b_2 = c_n$ and for every $i \in \{1, \dots, n\}$, we have either $(c_{i-1}, c_i) \in R$ or $(c_i, c_{i-1}) \in R$. We may assume that $(b_1, b_2) \in R$, because otherwise we may proceed inductively. Since $(b_1, b_2) \in R$, there is $a \in A$ such that $b_1 = f(a)$ and $b_2 = g(a)$. Then we have

$$z(b_1) = z(f(a)) = z(g(a)) = z(b_2),$$

hence $(b_1, b_2) \in \text{Ker}(z)$. Thus, we have $\text{Ker}(\pi) = \bar{R} \subseteq \text{Ker}(z)$. Using the factorization theorem of the function z by the surjective function π , there is a unique function $u : B/\bar{R} \rightarrow Z$ such that $u \circ \pi = z$.

3.4.2 The Category Ab

The following theorem will be useful.

Theorem 3.14 (Factorization Theorem by an Epimorphism). Let $f : A \rightarrow B$ be a homomorphism of abelian groups, and let $g : A \rightarrow A'$ be an epimorphism of abelian groups with $\text{Ker}(g) \subseteq \text{Ker}(f)$. Then there exists a unique homomorphism of abelian groups $h : A' \rightarrow B$ such that $f = h \circ g$, that is, the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{g} & A' \\ & \searrow f & \downarrow h \\ & & B \end{array}$$

Proof. Let $a' \in A'$. Since g is an epimorphism, there exists $a \in A$ such that $g(a) = a'$. If there exists $a_0 \in A$ such that $g(a_0) = a'$, then we have $g(a) = g(a_0)$. It follows that $g(a - a_0) = 0$, hence $a - a_0 \in \text{Ker}(g) \subseteq \text{Ker}(f)$. Then $f(a - a_0) = 0$, hence $f(a) = f(a_0)$.

It follows that we can define the function $h : A' \rightarrow B$ by $h(a') = f(a)$, where $f(a)$ is uniquely determined as above. We have $h(g(a)) = a$ for every $a \in A$. Hence $f = h \circ g$.

We show that h is a homomorphism of abelian groups. Let $a'_1, a'_2 \in A'$. Then there exist $a_1, a_2 \in A$ such that $g(a_1) = a'_1$ and $g(a_2) = a'_2$. Hence $h(a'_1) = f(a_1)$ and $h(a'_2) = f(a_2)$. We have

$$g(a_1 + a_2) = g(a_1) + g(a_2) = a'_1 + a'_2.$$

It follows that

$$h(a'_1 + a'_2) = h(g(a_1 + a_2)) = f(a_1 + a_2) = f(a_1) + f(a_2) = h(a'_1) + h(a'_2)$$

Thus h is a homomorphism of abelian groups.

For uniqueness, suppose that there exists a homomorphism $h' : A' \rightarrow B$ such that $f = h' \circ g$. It follows that $h \circ g = h' \circ g$. Since g is an epimorphism, we have $h = h'$. \square

Let $(A, +)$ and $(B, +)$ be abelian groups and let $f, g : A \rightarrow B$ be group homomorphisms. Since $I = \text{Im}(f - g)$ is a subgroup of B , we may consider the factor group

$$Q = B/I = \{b + I \mid b \in B\}$$

where the operation is defined by

$$(b_1 + I) + (b_2 + I) = (b_1 + b_2) + I, \quad \forall b_1, b_2 \in B$$

The factor group $B/\text{Im}(f - g)$ is also called a cokernel of the group homomorphism $f - g$, and we denote it by $\text{Coker}(f - g)$.

Consider the group homomorphism $q : B \rightarrow Q = B/I$ defined by $q(b) = b + I$. We have $\text{Ker}(q) = \{b \in B \mid q(b) = I\} = \{b \in B \mid b + I = I\} = I$.

We show that (Q, q) is a coequalizer of f and g .

For every $a \in A$ we have $f(a) - g(a) = (f - g)(a) \in I$, hence $f(a) + I = g(a) + I$. Then for every $a \in A$ we have

$$(q \circ f)(a) = f(a) + I = g(a) + I = (q \circ g)(a),$$

hence $q \circ f = q \circ g$.

Now let $(Z, +)$ be an abelian group and let $z : B \rightarrow Z$ be a group homomorphism such that $z \circ f = z \circ g$. We show that $I \subseteq \text{Ker}(z)$. To this end, let $b \in I$. Then $b = (f - g)(a)$ for some $a \in A$. It follows that

$$z(b) = z((f - g)(a)) = z(f(a) - g(a)) = z(f(a)) - z(g(a)) = 0,$$

hence $b \in \text{Ker}(z)$. Thus we have $\text{Ker}(q) = I \subseteq \text{Ker}(z)$. Using the factorization theorem for the group homomorphism z by the epimorphism q , there is a unique group homomorphism $u : B/I \rightarrow Z$ such that $u \circ q = z$.

In this category the coequalizer of two group homomorphisms is in fact the cokernel of some group homomorphism, namely

$$Q = \text{Coker}(f - g)$$

On the other hand, the cokernel of a group homomorphism is the coequalizer of some group homomorphism, namely

$$\text{Coker}(f) = B/\text{Im}(f) = B/\text{Im}(f - 0)$$

is the coequalizer of the group homomorphism $f : A \rightarrow B$ and the zero group homomorphism $0 : A \rightarrow B$.

3.4.3 Monoid Categories

Let (M, \cdot) be a monoid, which may be viewed as a monoid category. A coequalizer of two elements $a, b \in M$ is an element $x \in M$ with $xa = xb$ and for every $z \in M$ such that $za = zb$ there is $u \in M$ such that $ux = z$. If (M, \cdot) is a non-trivial group, then $a, b \in M$ with $a \neq b$ do not have a coequalizer, because there is no $x \in M$ such that $xa = xb$.

Definition 3.7. We say that a category **has coequalizers** if every arrows $f, g : A \rightarrow B$ have a coequalizer.

Example 3.3. The categories **Set** and **Ab** have coequalizers, while monoid categories may not have coequalizers.

Chapter 4

Limits and Colimits

4.1 Subobjects

Definition 4.1. Let X be an object of a category \mathcal{C} . A **subobject** of X is a monomorphism $m : M \rightarrow X$.

Given two subobjects $m : M \rightarrow X$ and $m' : M' \rightarrow X$, we define an arrow $f : m \rightarrow m'$ as an arrow $f : M \rightarrow M'$ such that $m' \circ f = m$, that is, the following diagram is commutative:

$$\begin{array}{ccc}
 M & \overset{f}{\dashrightarrow} & M' \\
 \searrow m & & \swarrow m' \\
 & X &
 \end{array}$$

In this way, we obtain a category whose objects are subobjects of a given object X and whose arrows are defined as above. We denote this category by $\text{Sub}_{\mathcal{C}}(X)$.

Suppose that there is an arrow $g : M \rightarrow M'$ such that $m' \circ g = m$. We also have $m' \circ f = m$. Then $m' \circ f = m' \circ g$, which implies $f = g$, because m' is a monomorphism. Hence there is a unique arrow between two subobjects.

We may view $\text{Sub}_{\mathcal{C}}(X)$ as a poset category. We define the following relation between the objects of $\text{Sub}_{\mathcal{C}}(X)$:

$$m \leq m' \iff \exists f : m \rightarrow m'$$

We say that m and m' are equivalent, and denote it by $m \equiv m'$, if they are isomorphic as subobjects, that is, $m \leq m'$ and $m' \leq m$. This means that there are arrows $f : M \rightarrow M'$ and $f' : M' \rightarrow M$ such that $m' \circ f = m$ and $m \circ f' = m'$. It follows that $m = m' \circ f = m \circ f' \circ f$, which implies that $f' \circ f = 1_M$, because m is a monomorphism. Similarly, one has $f \circ f' = 1_{M'}$. Hence f is an isomorphism and $M \cong M'$. This shows that equivalent subobjects have isomorphic domains.

We sometimes abuse notation and language by calling M the subobject when the monomorphism $m : M \rightarrow X$ is clear.

It is convenient to pass from the poset $(\text{Sub}_{\mathcal{C}}(X), \leq)$ to the poset obtained by factoring out the equivalence relation “ \equiv ”. In this way, a subobject is an equivalence class of monomorphisms under mutual inclusion.

If $m \leq m'$, then the arrow $f : M \rightarrow M'$ such that $m' \circ f = m$ is also a monomorphism, hence M is a subobject of M' . In this way we have a functor

$$\text{Sub}(M') \rightarrow \text{Sub}(X)$$

defined by composition by f . Note that the composite of monomorphisms is also a monomorphism.

4.2 Pullbacks

Definition 4.2. Let \mathcal{C} be a category and let

$$\begin{array}{ccc} & & B \\ & & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$$

be arrows in \mathcal{C} (that is, a cospan). A **pullback** (or **fibered product** or **cartesian square**) of f and g consists of arrows in \mathcal{C}

$$\begin{array}{ccc} P & \xrightarrow{p_2} & B \\ p_1 \downarrow & & \\ A & & \end{array}$$

(that is, a span) such that $f \circ p_1 = g \circ p_2$ with the following universal mapping property: given any object Z and any arrows $z_1 : Z \rightarrow A$ and $z_2 : Z \rightarrow B$ in \mathcal{C} such that $f \circ z_1 = g \circ z_2$, there is a unique arrow $u : Z \rightarrow P$ such that $p_1 \circ u = z_1$ and $p_2 \circ u = z_2$, hence the following diagram is commutative:

$$\begin{array}{ccccc} Z & & \xrightarrow{z_2} & & B \\ & \searrow u & & \searrow p_2 & \\ & & P & \xrightarrow{p_2} & B \\ & & p_1 \downarrow & & \downarrow g \\ & & A & \xrightarrow{f} & C \\ & \swarrow z_1 & & \swarrow f & \\ & & & & \end{array}$$

A pullback of arrows $f : A \rightarrow C$ and $g : B \rightarrow C$ in \mathcal{C} is denoted by (P, p_1, p_2) or $A \times_C B$.
Dually, we define pushouts.

Definition 4.3. Let \mathcal{C} be a category and let

$$\begin{array}{ccc} C & \xrightarrow{g} & B \\ f \downarrow & & \\ A & & \end{array}$$

be arrows in \mathcal{C} (that is, a span). A **pushout** (or **fibered coproduct** or **cocartesian square**) of f and g consists of arrows in \mathcal{C}

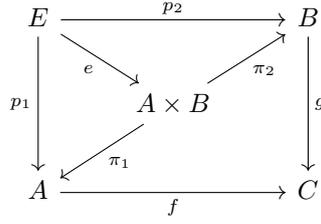
$$\begin{array}{ccc} & & B \\ & & \downarrow q_2 \\ A & \xrightarrow{q_1} & Q \end{array}$$

(that is, a cospan) such that $q_1 \circ f = q_2 \circ g$ with the following universal mapping property: given any object Z and any arrows $z_1 : A \rightarrow Z$ and $z_2 : B \rightarrow Z$ in \mathcal{C} such that $z_1 \circ f = z_2 \circ g$, there is a unique arrow $u : Q \rightarrow Z$ such that $u \circ q_1 = z_1$ and $u \circ q_2 = z_2$, hence the following diagram is commutative:

$$\begin{array}{ccccc} C & \xrightarrow{g} & B & & \\ f \downarrow & & \downarrow q_2 & & \\ A & \xrightarrow{q_1} & Q & \xrightarrow{z_2} & Z \\ & & \searrow u & & \\ & & & & \end{array}$$

A pushout of arrows $f : C \rightarrow A$ and $g : C \rightarrow B$ in \mathcal{C} is denoted by (Q, q_1, q_2) or $A \oplus_C B$.

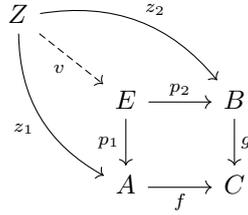
Theorem 4.1. Let \mathcal{C} be a category with binary products and equalizers. Consider the arrows $f : A \rightarrow C$ and $g : B \rightarrow C$ and the following diagram



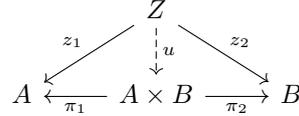
where π_1, π_2 are the canonical projections of the product $A \times B$, e is an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$, and $p_1 = \pi_1 \circ e$ and $p_2 = \pi_2 \circ e$. Then (E, p_1, p_2) is a pullback of f and g .

Proof. Since e is an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$ we have $f \circ \pi_1 \circ e = g \circ \pi_2 \circ e$. It follows that $f \circ p_1 = f \circ \pi_1 \circ e = g \circ \pi_2 \circ e = g \circ p_2$.

Now let Z be an object and $z_1 : Z \rightarrow A$ and $z_2 : Z \rightarrow B$ arrows in \mathcal{C} such that $f \circ z_1 = g \circ z_2$. We look for a unique arrow $v : Z \rightarrow E$ such that $p_1 \circ v = z_1$ and $p_2 \circ v = z_2$, that is, the following diagram is commutative:

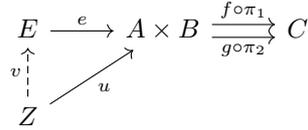


Consider the following diagram



By universal mapping property of the product $(A \times B, \pi_1, \pi_2)$, there is a unique arrow $u : Z \rightarrow A \times B$ such that $\pi_1 \circ u = z_1$ and $\pi_2 \circ u = z_2$.

We have $f \circ \pi_1 \circ u = f \circ z_1 = g \circ z_2 = g \circ \pi_2 \circ u$. Now consider the following diagram



By universal mapping property of the equalizer (E, e) , there is a unique arrow $v : Z \rightarrow E$ such that $e \circ v = u$. Then we have $p_1 \circ v = \pi_1 \circ e \circ v = \pi_1 \circ u = z_1$ and $p_2 \circ v = \pi_2 \circ e \circ v = \pi_2 \circ u = z_2$.

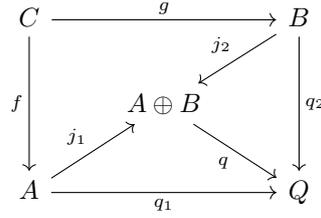
For uniqueness, suppose that there is also an arrow $v' : Z \rightarrow E$ such that $p_1 \circ v' = z_1$ and $p_2 \circ v' = z_2$. It follows that:

$$\begin{aligned}
 \pi_1 \circ (e \circ v') &= p_1 \circ v' = z_1 = p_1 \circ v = \pi_1 \circ (e \circ v), \\
 \pi_2 \circ (e \circ v') &= p_2 \circ v' = z_2 = p_2 \circ v = \pi_2 \circ (e \circ v).
 \end{aligned}$$

By universal mapping property of the product, we have $e \circ v' = e \circ v$, and thus $v' = v$, since e is a monomorphism. \square

Dually, we have the following theorem.

Theorem 4.2. Let \mathcal{C} be a category with binary coproducts and coequalizers. Consider the arrows $f : C \rightarrow A$ and $g : C \rightarrow B$ and the following diagram



where j_1, j_2 are the canonical injections of the coproduct $A \oplus B$, q is a coequalizer of $j_1 \circ f$ and $j_2 \circ g$, and $q_1 = q \circ j_1$ and $q_2 = q \circ j_2$. Then (Q, q_1, q_2) is a pushout of f and g .

Definition 4.4. We say that a category has **pullbacks** if every arrows $f : A \rightarrow C$ and $g : B \rightarrow C$ have a pullback. Dually, we say that a category has pushouts if every arrows $f : C \rightarrow A$ and $g : C \rightarrow B$ have a pushout.

Corollary 4.3. If a category has binary products and equalizers, then it has pullbacks. Dually, if a category has binary coproducts and coequalizers, then it has pushouts.

Example 4.1. Let us see that the converses in the above corollary do not hold in general. Let (G, \cdot) be a non-trivial group. View it as a category with one object G , with the elements of G as the arrows and the composition given by the product of elements (recall monoid category). This category has pullbacks and pushouts ($x, y \in G$ have pullback and pushout given by x^{-1} and y^{-1}), but it does not have (binary) products, (binary) coproducts, equalizers or coequalizers.

Next we present some examples of pullbacks and pushouts in certain categories.

4.2.1 The Category Set

(1) **Pullbacks.** Let $f : A \rightarrow C$ and $g : B \rightarrow C$ be functions. We construct their pullback by using binary products and equalizers, as described above. Using the above notation, (E, e) is an equalizer of $f \circ \pi_1$ and $g \circ \pi_2$, where $\pi_1 : A \times B \rightarrow A$ and $\pi_2 : A \times B \rightarrow B$ are the canonical projections defined by $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. By the construction of an equalizer in **Set** we have

$$\begin{aligned} E &= \{(a, b) \in A \times B \mid (f \circ \pi_1)(a, b) = (g \circ \pi_2)(a, b)\} \\ &= \{(a, b) \in A \times B \mid f(a) = g(b)\} \end{aligned}$$

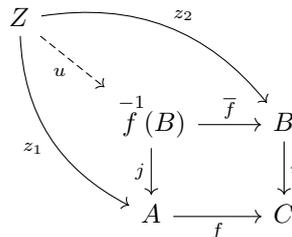
and $e : E \rightarrow A \times B$ is the inclusion function. By Theorem 4.1, (E, p_1, p_2) is a pullback of f and g , where $p_1 = \pi_1 \circ e : E \rightarrow A$ and $p_2 = \pi_2 \circ e : E \rightarrow B$ are given by

$$\begin{aligned} p_1(a, b) &= \pi_1(e(a, b)) = \pi_1(a, b) = a, \\ p_2(a, b) &= \pi_2(e(a, b)) = \pi_2(a, b) = b. \end{aligned}$$

Now let us analyze a particular pullback in **Set**. Let $f : A \rightarrow C$ be a function, $B \subseteq C$ and $i : B \rightarrow C$ the inclusion function. Consider the inverse image of B through f , that is,

$$\bar{f}^{-1}(B) = \{a \in A \mid f(a) \in B\}$$

Also, define $\bar{f} : \bar{f}^{-1}(B) \rightarrow B$ by $\bar{f} = f|_{\bar{f}^{-1}(B)}$, and let $j : \bar{f}^{-1}(B) \rightarrow A$ be the inclusion function. Let us show that $(\bar{f}^{-1}(B), j, \bar{f})$ is a pullback of f and i .



If $a \in \bar{f}^{-1}(B)$, then $f(a) \in B$, and thus $\bar{f}(a) \in B$. We have $i \circ \bar{f} = f \circ j$. Now let $z_1 : Z \rightarrow A$ and $z_2 : Z \rightarrow B$ be functions such that $f \circ z_1 = i \circ z_2$. Define $u : Z \rightarrow \bar{f}^{-1}(B)$ by $u(x) = z_1(x)$. Since $f(z_1(x)) = i(z_2(x)) = z_2(x) \in B$, we have $z_1(x) \in \bar{f}^{-1}(B)$, and thus u is well defined. Then we have $j \circ u = z_1$. Also, for every $x \in Z$ we have

$$(\bar{f} \circ u)(x) = \bar{f}(z_1(x)) = f(z_1(x)) = i(z_2(x)) = z_2(x),$$

hence $\bar{f} \circ u = z_2$.

For uniqueness, suppose that there is also a function $u' : Z \rightarrow \bar{f}^{-1}(B)$ such that $j \circ u' = z_1$ and $\bar{f} \circ u' = z_2$. Then we have $j \circ u' = j \circ u$, hence $u' = u$, because j is a monomorphism.

(2) **Pushouts.** Let $f : C \rightarrow A$ and $g : C \rightarrow B$ be functions. We construct their pushout by using binary coproducts and coequalizers, as described above. Consider the coproduct of A and B , namely $(A \sqcup B, j_1, j_2)$, where $A \sqcup B = \{(a, 1) \mid a \in A\} \cup \{(b, 2) \mid b \in B\}$ is the disjoint union of A and B , and $j_1 : A \rightarrow A \sqcup B$ and $j_2 : B \rightarrow A \sqcup B$ are the canonical injections defined by $j_1(a) = (a, 1)$ and $j_2(b) = (b, 2)$. Consider the functions $j_1 \circ f, j_2 \circ g : C \rightarrow A \sqcup B$. As in the construction of a coequalizer in **Set**, consider the relation $r = (A \sqcup B, A \sqcup B, R)$, whose graph R contains all pairs $(a, 1), (b, 2)$ for which there is $c \in C$ such that $f(c) = a$ and $g(c) = b$. Let $\bar{r} = (A \sqcup B, A \sqcup B, \bar{R})$ be the smallest equivalence relation on $A \sqcup B$ containing r . Consider the partition $Q = (A \sqcup B)/\bar{R}$ and the function $q : A \sqcup B \rightarrow Q$ defined by $q(x) = \bar{R}(x)$. Then we know that (Q, q) is a coequalizer of $j_1 \circ f$ and $j_2 \circ g$. Let $q_1 = q \circ j_1 : A \rightarrow Q$ and $q_2 = q \circ j_2 : B \rightarrow Q$. Then (Q, q_1, q_2) is a pushout of f and g by Theorem 4.2.

4.2.2 Poset Categories

Let (L, \leq) be a lattice, which may be viewed as a poset category. The pullback of a pair of arrows $a \leq c$ and $b \leq c$ is $(\inf(a, b), p_1, p_2)$, where $p_1 : \inf(a, b) \rightarrow a$ and $p_2 : \inf(a, b) \rightarrow b$ are the unique arrows having the given domains and codomains. Note that it coincides with the product of a and b !

The pushout of a pair of arrows $c \leq a$ and $c \leq b$ is $(\sup(a, b), q_1, q_2)$, where $q_1 : a \rightarrow \sup(a, b)$ and $q_2 : b \rightarrow \sup(a, b)$ are the unique arrows having the given domains and codomains. Note that it coincides with the coproduct of a and b !

If a poset (L, \leq) is not a lattice, two elements of L might not have a pullback or a pushout.

Example 4.2. The category **Set** has pullbacks and pushouts, while poset categories may not have pullbacks and pushouts.

4.3 Properties of Pullbacks

One may prove the following property.

Proposition 4.4. Consider the following commutative diagram in a category:

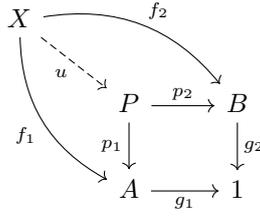
$$\begin{array}{ccccc} F & \xrightarrow{f'} & E & \xrightarrow{g'} & D \\ h'' \downarrow & & \downarrow h' & & \downarrow h \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

in which the right square is a pullback. Then the left square is a pullback if and only if the outer rectangle is a pullback.

Theorem 4.5. A category \mathcal{C} has finite products and equalizers if and only if \mathcal{C} has pullbacks and terminal object.

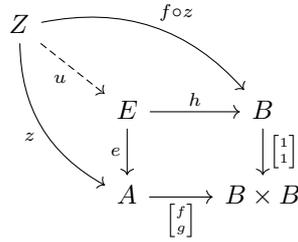
Proof. Assume that \mathcal{C} has finite products and equalizers. Then we have seen that \mathcal{C} has pullbacks. Also, note that the empty product is the terminal object.

Conversely, assume that \mathcal{C} has pullbacks and terminal object, denoted by 1 . Let A and B be objects of \mathcal{C} . Then there are unique arrows $f : A \rightarrow 1$ and $g : B \rightarrow 1$. Consider their pullback



We show that (P, p_1, p_2) is a product of A and B . Let $f_1 : X \rightarrow A$ and $f_2 : X \rightarrow B$ be arrows in \mathcal{C} . We have $g_1 \circ f_1, g_2 \circ f_2 : X \rightarrow 1$. Since 1 is a terminal object, we must have $g_1 \circ f_1 = g_2 \circ f_2$. By universal mapping property of the pullback there is a unique arrow $u : X \rightarrow P$ such that $p_1 \circ u = f_1$ and $p_2 \circ u = f_2$. This shows that (P, p_1, p_2) is a product of A and B .

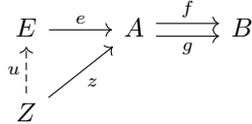
Now let $f, g : A \rightarrow B$ be arrows in \mathcal{C} . Consider the arrows $\begin{bmatrix} f \\ g \end{bmatrix} : A \rightarrow B \times B$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix} : B \rightarrow B \times B$. Consider their pullback



We show that (E, e) is an equalizer of f and g . We have $\begin{bmatrix} f \\ g \end{bmatrix} \cdot [e] = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot [h]$, whence $f \circ e = h = g \circ e$. Now let Z be an object and $z : Z \rightarrow A$ an arrow in \mathcal{C} such that $f \circ z = g \circ z$. It follows that

$$\begin{bmatrix} f \\ g \end{bmatrix} \cdot [z] = \begin{bmatrix} f \circ z \\ g \circ z \end{bmatrix} = \begin{bmatrix} f \circ z \\ f \circ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot [f \circ z].$$

By universal mapping property of the pullback there is a unique arrow $u : Z \rightarrow E$ such that $e \circ u = z$ and $h \circ u = f \circ z$.



For uniqueness, suppose that there is also an arrow $u' : Z \rightarrow E$ such that $e \circ u' = z$. Then $e \circ u = e \circ u'$, whence $\begin{bmatrix} f \\ g \end{bmatrix} \cdot [e \circ u] = \begin{bmatrix} f \\ g \end{bmatrix} \cdot [e \circ u']$. Hence

$$\begin{bmatrix} h \circ u \\ h \circ u \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot [h \circ u] = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot [h \circ u'] = \begin{bmatrix} h \circ u' \\ h \circ u' \end{bmatrix}.$$

It follows that $h \circ u = h \circ u'$, hence $f \circ z = f \circ e \circ u = h \circ u'$. Thus we have $e \circ u' = z$ and $h \circ u' = f \circ z$. By the uniqueness property of the pullback, we must have $u' = u$. This shows that (E, e) is an equalizer of f and g . \square

Dually, we have the following theorem.

Theorem 4.6. A category \mathcal{C} has finite coproducts and coequalizers if and only if \mathcal{C} has pushouts and initial object.

4.4 Limits

Definition 4.5. Let \mathcal{J} and \mathcal{C} be categories. A diagram of type \mathcal{J} in \mathcal{C} is a functor $D : \mathcal{J} \rightarrow \mathcal{C}$. We write the objects of \mathcal{J} by i, j etc. and we call \mathcal{J} the **index category**. We denote the values of the functor D in the form D_i, D_j etc.

A **cone** to a diagram D consists of an object C of \mathcal{C} and a family of arrows $c_j : C \rightarrow D_j$ in \mathcal{C} , one for each object $j \in \mathcal{J}$ such that for every arrow $\alpha : i \rightarrow j$ in \mathcal{J} , the following triangle is commutative:

$$\begin{array}{ccc} C & \xrightarrow{c_j} & D_j \\ c_i \downarrow & \nearrow D_\alpha & \\ D_i & & \end{array}$$

A **morphism of cones** (or **arrow of cones**) $\nu : (C, c_j) \rightarrow (C', c'_j)$ is an arrow in \mathcal{C} from C to C' making the following triangles commutative ($j \in \mathcal{J}$):

$$\begin{array}{ccc} C & \xrightarrow{\nu} & C' \\ c_j \downarrow & \swarrow c'_j & \\ D_j & & \end{array}$$

In this way we obtain a category $\text{Cone}(D)$ with:

- objects: the cones to D
- arrows: the morphisms of cones to D
- composition: composition of arrows in \mathcal{C} , compatible with commutative triangles
- identity arrow: the identity morphism of a cone

Remark 4.7. We are here thinking of the diagram D as a “picture of \mathcal{J} in \mathcal{C} ”. A cone to such a diagram D is then imagined as a many-sided pyramid over the “base” D , and a morphism of cones is an arrow between the apexes of such pyramids.

Definition 4.6. A **limit** for a diagram $D : \mathcal{J} \rightarrow \mathcal{C}$ is a terminal object in $\text{Cone}(D)$. A **finite limit** is a limit for a diagram on a finite index category \mathcal{J} .

A limit will be denoted by $p_i : \varprojlim D_j \rightarrow D_i$.

Proposition 4.8. The limit is uniquely determined up to isomorphism.

Remark 4.9. Detailing the definition, the limit of a diagram D has the following universal mapping property: given any cone (C, c_j) to D , there is a unique arrow $u : C \rightarrow \varprojlim D_j$ such that for all j we have $p_j \circ u = c_j$. Thus, the limiting cone $(\varprojlim D_j, p_j)$ can be thought of as the “closest” cone to the diagram D , and indeed any other cone (C, c_j) comes from it just by composing with an arrow at the vertex, namely $u : C \rightarrow \varprojlim D_j$.

$$\begin{array}{ccc} & C & \\ & \downarrow u & \\ c_i \swarrow & \varprojlim D_j & \searrow c_j \\ p_i \swarrow & & \searrow p_j \\ D_i & \xrightarrow{D_\alpha} & D_j \end{array}$$

Example 4.3. (1) Let \mathcal{J} be the category with two objects, denoted 1 and 2, and no nonidentity arrow:

$$1 \qquad 2$$

A diagram on \mathcal{J} is a functor $D : \mathcal{J} \rightarrow \mathcal{C}$, which takes the objects 1, 2 into objects D_1, D_2 .

A cone to D is an object C together with arrows $c_1 : C \rightarrow D_1$ and $c_2 : C \rightarrow D_2$. Hence it is a span as follows:

$$D_1 \xleftarrow{c_1} C \xrightarrow{c_2} D_2$$

A limit for D is a terminal object in $\text{Cone}(D)$, that is, for every cone to D (span)

$$D_1 \xleftarrow{c'_1} C' \xrightarrow{c'_2} D_2$$

there is a unique morphism between the two cones, that is, there is a unique arrow $u : C' \rightarrow C$ in \mathcal{C} such that $c_1 \circ u = c'_1$ and $c_2 \circ u = c'_2$.

$$\begin{array}{ccc} & C' & \\ c'_1 \swarrow & \downarrow u & \searrow c'_2 \\ D_1 & \xleftarrow{c_1} C \xrightarrow{c_2} & D_2 \end{array}$$

(2) Let \mathcal{J} be the category having two objects, denoted 1 and 2, and two non-identity arrows as follows:

$$1 \begin{array}{c} \xrightarrow{\alpha} \\ \xrightarrow{\beta} \end{array} 2$$

A diagram on \mathcal{J} is a functor $D : \mathcal{J} \rightarrow \mathcal{C}$, which takes an object i into an object D_i for $i = 1, 2$, and the arrows $\alpha, \beta : 1 \rightarrow 2$ into arrows $D_\alpha, D_\beta : D_1 \rightarrow D_2$.

A cone to D is an object C together with arrows $c_1 : C \rightarrow D_1$ and $c_2 : C \rightarrow D_2$ such that the following diagram is commutative:

$$\begin{array}{ccc} & C & \\ c_1 \swarrow & & \searrow c_2 \\ D_1 & \begin{array}{c} \xrightarrow{D_\alpha} \\ \xrightarrow{D_\beta} \end{array} & D_2 \end{array}$$

hence $D_\alpha \circ c_1 = c_2$ and $D_\beta \circ c_1 = c_2$, and thus $D_\alpha \circ c_1 = D_\beta \circ c_1$.

A limit for D is a terminal object in $\text{Cone}(D)$, that is,

$$\begin{array}{ccc} C & \xrightarrow{c_1} & D_1 \begin{array}{c} \xrightarrow{D_\alpha} \\ \xrightarrow{D_\beta} \end{array} & D_2 \\ \uparrow u & \nearrow c'_1 & & \\ C' & & & \end{array}$$

For every object C' and arrow $c'_1 : C' \rightarrow D_1$ in \mathcal{C} , there is a unique arrow $u : C' \rightarrow C$ such that $c_1 \circ u = c'_1$.

Hence the limit is an equalizer of D_α and D_β .

(3) Let $\mathcal{J} = \emptyset$. Then $\lim_{\leftarrow} D_j \cong 1$ is a terminal object.

(4) A limit on the following finite category \mathcal{J} will be a pullback:

$$\begin{array}{ccc} & & * \\ & & \downarrow \\ * & \longrightarrow & * \end{array}$$

Remark 4.10. Products, equalizers, terminal objects and pullbacks are all examples of finite limits.

Remark 4.11. Dually, one may define the notion of **colimit**, which will be a limit in the opposite category. A colimit will be denoted by $q_i : D_i \rightarrow \varinjlim D_j$. The colimit is uniquely determined up to an isomorphism. Coproducts, coequalizers, initial objects and pushouts are all examples of finite colimits.

One may get the following property and its dual.

Theorem 4.12. A category has all finite limits if and only if it has finite products and equalizers (if and only if it has pullbacks and a terminal object).

Theorem 4.13. A category has all finite colimits if and only if it has finite coproducts and coequalizers (if and only if it has pushouts and an initial object).

4.5 Preservation of Limits

Definition 4.7. A covariant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is said to **preserve limits of type \mathcal{J}** if whenever $p_j : L \rightarrow D_j$ is a limit for a diagram $D : \mathcal{J} \rightarrow \mathcal{C}$, $F(p_j) : F(L) \rightarrow F(D_j)$ is a limit for a diagram $F \circ D : \mathcal{J} \rightarrow \mathcal{D}$.

Briefly, we write

$$F(\varprojlim D_j) \cong \varprojlim F(D_j).$$

One has the following results.

Theorem 4.14. Let \mathcal{C} be a locally small category and let A be an object of \mathcal{C} . Then the covariant representable functor $\text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathbf{Set}$ preserves all limits, that is,

$$\text{Hom}_{\mathcal{C}}(A, \varprojlim D_j) \cong \varprojlim \text{Hom}_{\mathcal{C}}(A, D_j).$$

Theorem 4.15. Let \mathcal{C} be a locally small category and let A be an object of \mathcal{C} . Then the contravariant representable functor $\text{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \rightarrow \mathbf{Set}$ maps all colimits to limits, that is,

$$\text{Hom}_{\mathcal{C}}(\varinjlim D_j, A) \cong \varprojlim \text{Hom}_{\mathcal{C}}(D_j, A).$$

Chapter 5

Naturality

5.1 Fully Faithful Functors

Definition 5.1. Let \mathcal{C} and \mathcal{D} be locally small categories. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor and for every objects A, B of \mathcal{C} consider the map

$$F_{A,B} : \text{Hom}_{\mathcal{C}}(A, B) \rightarrow \text{Hom}_{\mathcal{D}}(F(A), F(B)), \quad F_{A,B}(f) = F(f).$$

Then F is called:

- (1) **faithful** if for every objects A, B of \mathcal{C} the map $F_{A,B}$ is injective.
- (2) **full** if for every objects A, B of \mathcal{C} the map $F_{A,B}$ is surjective.
- (3) **fully faithful** if it is full and faithful, that is, for every objects A, B of \mathcal{C} the map $F_{A,B}$ is bijective.

Definition 5.2. Let \mathcal{C} be a category with the class of objects \mathcal{C}_0 and the class of arrows \mathcal{C}_1 . By a subcategory of \mathcal{C} we mean a collection \mathcal{U} of some objects $\mathcal{U}_0 \subseteq \mathcal{C}_0$ and some arrows $\mathcal{U}_1 \subseteq \mathcal{C}_1$ closed under taking composition, under taking domains and codomains, and under taking identity. Hence there is a functor $i : \mathcal{U} \rightarrow \mathcal{C}$ called the **inclusion functor**.

By a **full subcategory** of \mathcal{C} we mean a collection \mathcal{U} of some objects of \mathcal{C} together with all the arrows between them. Hence the inclusion functor $i : \mathcal{U} \rightarrow \mathcal{C}$ is fully faithful in this case.

Example 5.1. (1) The inclusion functor $i : \mathbf{Set}_{fin} \rightarrow \mathbf{Set}$ is fully faithful, where \mathbf{Set}_{fin} is the category of finite sets and functions. Note that the map

$$i_{A,B} : \text{Hom}_{\mathbf{Set}_{fin}}(A, B) \rightarrow \text{Hom}_{\mathbf{Set}}(i(A), i(B)), \quad i_{A,B}(f) = i(f) = f$$

is bijective. The category \mathbf{Set}_{fin} is a full subcategory of \mathbf{Set} .

(2) The **forgetful** functor $U : \mathbf{Grp} \rightarrow \mathbf{Set}$ is defined on objects (groups) (G, \cdot) of \mathbf{Grp} by the underlying set G of (G, \cdot) , and on arrows (group homomorphisms) $f : A \rightarrow B$ in \mathbf{Grp} by the underlying function $f : A \rightarrow B$. Then U is faithful, but not full. Note that for every groups (A, \cdot) and (B, \cdot) , the map

$$U_{A,B} : \text{Hom}_{\mathbf{Grp}}(A, B) \rightarrow \text{Hom}_{\mathbf{Set}}(U(A), U(B)), \quad U_{A,B}(f) = U(f) = f$$

is injective, but not surjective. If $U_{A,B}$ were surjective, then for every function $\varphi : U(A) \rightarrow U(B)$, there would exist a group homomorphism $f : A \rightarrow B$ such that $f = U_{A,B}(f) = \varphi$, which is false. For instance, any function between two groups which does not preserve the identity element cannot be a group homomorphism.

(3) The inclusion functor $i : \mathbf{Ring} \rightarrow \mathbf{Rng}$ is clearly faithful, but not full. Note that there are non-unitary homomorphisms in \mathbf{Rng} between unitary rings, and thus they are not arrows in \mathbf{Ring} . For instance, if R is any unitary ring, then $|\text{Hom}_{\mathbf{Ring}}(\mathbb{Z}, R)| = 1$, but $|\text{Hom}_{\mathbf{Rng}}(\mathbb{Z}, R)| > 1$ (the trivial function $\mathbb{Z} \rightarrow R$ is a non-unitary homomorphisms in \mathbf{Rng}), hence the map

$$i_{\mathbb{Z},R} : \text{Hom}_{\mathbf{Ring}}(\mathbb{Z}, R) \rightarrow \text{Hom}_{\mathbf{Rng}}(i(\mathbb{Z}), i(R)), \quad i_{\mathbb{Z},R}(f) = i(f) = f$$

cannot be surjective. The category **Ring** is a subcategory of **Rng**, but not a full subcategory.

(4) Let (M, \cdot) and (N, \cdot) be monoids, and view them as categories with one object. Note that a functor between monoid categories is just a monoid homomorphism. Moreover, a faithful functor is the same as an injective monoid homomorphism, while a full functor is the same as a surjective monoid homomorphism. These offer immediate examples of functors having both, one or none of the properties of being faithful or full. For instance, every surjective monoid homomorphism which is not injective gives an example of a full functor which is not faithful.

(5) Let \mathcal{C} and \mathcal{D} be locally small categories, and let D be an object of \mathcal{D} . The constant functor $F : \mathcal{C} \rightarrow \mathcal{D}$ determined by D is defined on objects by $F(C) = D$, and on arrows by $F(f) = 1_D$. In general, the constant functor is neither faithful, nor full (if \mathcal{C} and \mathcal{D} have each at least two arrows).

Lemma 5.1. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a fully faithful functor. Then F is “essentially injective” on objects in the sense that $F(A) \cong F(B)$ implies $A \cong B$.

Proof. Let $g : F(A) \rightarrow F(B)$ be an isomorphism. Consider $g^{-1} : F(B) \rightarrow F(A)$. Since F is fully faithful, there are (unique) arrows $f : A \rightarrow B$ such that $F(f) = g$ and $f' : B \rightarrow A$ such that $F(f') = g^{-1}$. We have $F(f \circ f') = F(f) \circ F(f') = g \circ g^{-1} = 1_{F(B)} = F(1_B)$, which implies that $f \circ f' = 1_B$, because F is fully faithful. Similarly, we have $f' \circ f = 1_A$. Hence $f : A \rightarrow B$ is an isomorphism. Thus, F is essentially injective on objects. \square

Remark 5.2. (1) A fully faithful functor need not be injective on objects. For instance, let \mathcal{C} be a category with two objects C_1 and C_2 , and two non-trivial morphisms $f : C_1 \rightarrow C_2$ and $g : C_2 \rightarrow C_1$ such that $g \circ f = 1_{C_1}$ and $f \circ g = 1_{C_2}$, and let \mathcal{D} be a category with one object D and one morphism 1_D . Then the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ given by $F(C_1) = F(C_2) = D$ and $F(f) = 1_D$ is fully faithful, but it is not injective on objects.

(2) A fully faithful functor need not be “essentially surjective” on objects in the sense that for every object D of \mathcal{D} , there is some object C of \mathcal{C} such that $F(C) \cong D$. For instance, the inclusion functor $i : \mathbf{Ab} \rightarrow \mathbf{Grp}$ is fully faithful, but not essentially surjective on objects.

Definition 5.3. Let \mathcal{C} be a locally small category. An object C of \mathcal{C} is called a:

- (1) **generator** of \mathcal{C} if the covariant functor $\text{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \rightarrow \mathbf{Set}$ is faithful.
- (2) **cogenerator** of \mathcal{C} if the contravariant functor $\text{Hom}_{\mathcal{C}}(-, C) : \mathcal{C} \rightarrow \mathbf{Set}$ is faithful.

Proposition 5.3. Let \mathcal{C} be a locally small category, and let C be an object of \mathcal{C} . Then:

- (1) C is a generator of \mathcal{C} if and only if for every arrows $f, g : X \rightarrow Y$ in \mathcal{C} with $f \neq g$, there is an arrow $c : C \rightarrow X$ in \mathcal{C} such that $f \circ c \neq g \circ c$.
- (2) C is a cogenerator of \mathcal{C} if and only if for every arrows $f, g : X \rightarrow Y$ in \mathcal{C} with $f \neq g$, there is an arrow $c : Y \rightarrow C$ in \mathcal{C} such that $c \circ f \neq c \circ g$.

Proof. (1) The object C is a generator of \mathcal{C} if and only if the functor $\text{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \rightarrow \mathbf{Set}$ is faithful if and only if for every arrows $f, g : X \rightarrow Y$ in \mathcal{C} with $f \neq g$, we have $\text{Hom}_{\mathcal{C}}(C, f) \neq \text{Hom}_{\mathcal{C}}(C, g)$ if and only if for every arrows $f, g : X \rightarrow Y$ in \mathcal{C} with $f \neq g$, there is an arrow $c : C \rightarrow X$ in \mathcal{C} such that $\text{Hom}_{\mathcal{C}}(C, f)(c) \neq \text{Hom}_{\mathcal{C}}(C, g)(c)$ if and only if for every arrows $f, g : X \rightarrow Y$ in \mathcal{C} with $f \neq g$, there is an arrow $c : C \rightarrow X$ in \mathcal{C} such that $f \circ c \neq g \circ c$. \square

Example 5.2. (1) The terminal object 1 is a generator of **Set**. Recall that 1 is any single element set, say $\{a\}$. Let X, Y be sets and $f, g : X \rightarrow Y$ functions such that $f \neq g$. Then there is $x_0 \in X$ such that $f(x_0) \neq g(x_0)$. We look for a function $c : \{a\} \rightarrow X$ such that $f \circ c \neq g \circ c$. We have $f \circ c \neq g \circ c$ if and only if $f(c(a)) \neq g(c(a))$. Just take $c(a) = x_0$. Actually, by a similar reasoning, it is easy to see that any non-empty set is a generator of **Set**.

Every set C with $|C| \geq 2$ is a cogenerator of **Set**. Let $a, b \in C$. Let X, Y be sets and $f, g : X \rightarrow Y$ functions such that $f \neq g$. Then there is $x_0 \in X$ such that $f(x_0) \neq g(x_0)$, and thus $|Y| \geq 2$. We look for a function $c : Y \rightarrow C$ such that $c \circ f \neq c \circ g$. We have $c \circ f \neq c \circ g$ if and only if there is $x \in X$ such that $c(f(x)) \neq c(g(x))$. We may define $c : Y \rightarrow C$ by $c(f(x_0)) = a$ and $c(y) = b$ for every $y \neq f(x_0)$. In particular, $c(g(x_0)) = b$ and $c(f(x_0)) = a \neq b = c(g(x_0))$. Hence C is a cogenerator of **Set** by the dual of Proposition 5.3.

(2) The field K is a generator of the category $\mathbf{Vect}(K)$. We show that the covariant functor $F = \text{Hom}_K(K, -) : \mathbf{Vect}(K) \rightarrow \mathbf{Set}$ is faithful. We claim that

$$F(V) = \text{Hom}_K(K, V) \cong V, \quad \forall V \in \mathbf{Vect}(K).$$

To this end, define $\varphi : \text{Hom}_K(K, V) \rightarrow V$ by $\varphi(f) = f(1)$, and $\psi : V \rightarrow \text{Hom}_K(K, V)$ by $\psi(v) = t_v$, where $t_v : K \rightarrow V$ is given by $t_v(k) = kv$. Then φ and ψ are K -linear maps. We have $\varphi(\psi(v)) = \varphi(t_v) = t_v(1) = v$ for every $v \in V$, hence $\varphi \circ \psi = 1_V$. Also, we have $\psi(\varphi(f)) = \psi(f(1)) = t_{f(1)} = f$ for every $f \in \text{Hom}_K(K, V)$, hence $\psi \circ \varphi = 1_{\text{Hom}_K(K, V)}$. Therefore, φ is a K -isomorphism. Now it is clear that for every K -vector spaces A and B , the map

$$F_{A,B} : \text{Hom}_{\mathbf{Vect}(K)}(A, B) \rightarrow \text{Hom}_{\mathbf{Set}}(F(A), F(B)), \quad F_{A,B}(f) = \text{Hom}_K(K, f)$$

is injective, hence $F = \text{Hom}_K(K, -) : \mathbf{Vect}(K) \rightarrow \mathbf{Set}$ is faithful. Hence K is a generator of $\mathbf{Vect}(K)$.

One may also show that K is a cogenerator of the category $\mathbf{Vect}(K)$.

5.2 Naturality

Example 5.3. Let \mathcal{C} be a category with binary products. For every objects A, B, C of \mathcal{C} we have an isomorphism $h : (A \times B) \times C \cong A \times (B \times C)$. This is an isomorphism regardless of what objects A, B, C are. For every arrow $f : A \rightarrow A'$ in \mathcal{C} , we have the following commutative diagram

$$\begin{array}{ccc} (A \times B) \times C & \xrightarrow{h_A} & A \times (B \times C) \\ (f \times 1_B) \times 1_C \downarrow & & \downarrow f \times (1_B \times 1_C) \\ (A' \times B) \times C & \xrightarrow{h'_A} & A' \times (B \times C) \end{array}$$

where h_A and $h_{A'}$ are isomorphisms. In fact, we have an isomorphism between “constructions”:

$$(- \times B) \times C \cong - \times (B \times C).$$

Actually, this is an “isomorphism of functors”.

Definition 5.4. Let $F, G : \mathcal{C} \rightarrow \mathcal{D}$ be functors. By a natural transformation between F and G , denoted by $\nu : F \rightarrow G$, we mean a family $(\nu_C : F(C) \rightarrow G(C))_{C \in \mathcal{C}}$ of arrows in \mathcal{D} such that for every arrow $f : C \rightarrow C'$ in \mathcal{C} we have the following commutative diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{\nu_C} & G(C) \\ F(f) \downarrow & & \downarrow G(f) \\ F(C') & \xrightarrow{\nu_{C'}} & G(C') \end{array}$$

hence $G(f) \circ \nu_C = \nu_{C'} \circ F(f)$. Here ν_C is called the **component** of ν at C .

Example 5.4. Let \mathcal{C} be the category \mathbf{Set} . Consider the functor $F = \times : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ defined on objects (A, B) of $\mathcal{C} \times \mathcal{C}$ by $F(A, B) = A \times B$, and on arrows (α, β) in $\mathcal{C} \times \mathcal{C}$ by $F(\alpha, \beta) = \alpha \times \beta$. Also, consider the functor $G = \bar{\times} : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ defined on objects (A, B) of $\mathcal{C} \times \mathcal{C}$ by $G(A, B) = B \times A$, and on arrows (α, β) in $\mathcal{C} \times \mathcal{C}$ by $G(\alpha, \beta) = \beta \times \alpha$.

Define a “twist” natural transformation whose component at an object (A, B) of $\mathcal{C} \times \mathcal{C}$ is

$$t_{(A,B)} : F(A, B) = A \times B \rightarrow G(A, B) = B \times A, \quad t_{(A,B)}(a, b) = (b, a).$$

Then for every arrow (α, β) in $\mathcal{C} \times \mathcal{C}$ consider the following diagram:

$$\begin{array}{ccc} F(A, B) = A \times B & \xrightarrow{t_{(A,B)}} & G(A, B) = B \times A \\ F(\alpha, \beta) = \alpha \times \beta \downarrow & & \downarrow G(\alpha, \beta) = \beta \times \alpha \\ F(A', B') = A' \times B' & \xrightarrow{t_{(A',B')}} & G(A', B') = B' \times A' \end{array}$$

The diagram is commutative, because we have:

$$\begin{aligned} [(\beta \times \alpha) \circ t_{(A,B)}](a, b) &= (\beta \times \alpha)(b, a) = (\beta(b), \alpha(a)) \\ &= t_{(A',B')}(\alpha(a), \beta(b)) \\ &= [t_{(A',B')} \circ (\alpha \times \beta)](a, b) \end{aligned}$$

for every $(a, b) \in A \times B$. Hence $t : F \rightarrow G$ is a natural transformation. Here all $t_{(A,B)}$ are bijections (isomorphisms in **Set**).

Proposition 5.4. Let $F, G, H : \mathcal{C} \rightarrow \mathcal{D}$ be functors. Let $\nu : F \rightarrow G$ and $\Phi : G \rightarrow H$ be natural transformations. Then the family $(\Phi_C \circ \nu_C : F(C) \rightarrow H(C))_{C \in \mathcal{C}}$ of arrows in \mathcal{D} defines a natural transformation, denoted by $\Phi \circ \nu : F \rightarrow H$ and called the composite of the natural transformations Φ and ν .

Proof. For every arrow $f : C \rightarrow C'$ in \mathcal{C} consider the following diagram:

$$\begin{array}{ccccc} F(C) & \xrightarrow{\nu_C} & G(C) & \xrightarrow{\Phi_C} & H(C) \\ F(f) \downarrow & & \downarrow G(f) & & \downarrow H(f) \\ F(C') & \xrightarrow{\nu_{C'}} & G(C') & \xrightarrow{\Phi_{C'}} & H(C') \end{array}$$

Since $\nu : F \rightarrow G$ and $\Phi : G \rightarrow H$ are natural transformations, the two squares are commutative. Then the outer rectangle is also commutative. Hence the family $((\Phi \circ \nu)_C = \Phi_C \circ \nu_C : F(C) \rightarrow H(C))_{C \in \mathcal{C}}$ of arrows in \mathcal{D} defines a natural transformation $\Phi \circ \nu : F \rightarrow H$. \square

Definition 5.5. Let \mathcal{C} and \mathcal{D} be categories. The **functor category** $\text{Fun}(\mathcal{C}, \mathcal{D})$ is defined by:

- objects: the (covariant) functors $F : \mathcal{C} \rightarrow \mathcal{D}$
- arrows: the natural transformations between functors from \mathcal{C} to \mathcal{D}
- composition: the composite of natural transformations
- identity arrow: for every object F of $\text{Fun}(\mathcal{C}, \mathcal{D})$, the identity arrow is the natural transformation $1_F : F \rightarrow F$ with components $(1_F)_C = 1_{F(C)} : F(C) \rightarrow F(C)$ for every object C of \mathcal{C} .

Definition 5.6. A **natural isomorphism** is a natural transformation which is an isomorphism in $\text{Fun}(\mathcal{C}, \mathcal{D})$. We denote a natural isomorphism between functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$ by $F \cong G$.

Proposition 5.5. A natural transformation $\alpha : F \rightarrow G$ is a natural isomorphism if and only if each component $\alpha_C : F(C) \rightarrow G(C)$ is an isomorphism.

Proof. Let $\alpha : F \rightarrow G$ be a natural transformation between two functors $F, G : \mathcal{C} \rightarrow \mathcal{D}$.

Suppose that $\alpha : F \rightarrow G$ is a natural isomorphism. Then it has an inverse $\alpha^{-1} : G \rightarrow F$. Hence $\alpha^{-1} \circ \alpha = 1_F$ and $\alpha \circ \alpha^{-1} = 1_G$. Then for every object C of \mathcal{C} , we have $(\alpha^{-1})_C \circ \alpha_C = (\alpha^{-1} \circ \alpha)_C = 1_{F(C)}$ and $\alpha_C \circ (\alpha^{-1})_C = (\alpha \circ \alpha^{-1})_C = 1_{G(C)}$, which imply that α_C is an isomorphism.

Conversely, suppose that each component $\alpha_C : F(C) \rightarrow G(C)$ is an isomorphism. For every object C of \mathcal{C} define $\alpha_C^{-1} = (\alpha_C)^{-1}$. Let us show that this defines a natural transformation $\alpha^{-1} : G \rightarrow F$. To this end, let $f : C \rightarrow C'$ be an arrow in \mathcal{C} . We show that the following diagram is commutative:

$$\begin{array}{ccc} G(C) & \xrightarrow{\alpha_C^{-1}} & F(C) \\ G(f) \downarrow & & \downarrow F(f) \\ G(C') & \xrightarrow{\alpha_{C'}^{-1}} & F(C') \end{array}$$

Since $\alpha : F \rightarrow G$ is a natural transformation, we have the commutative diagram:

$$\begin{array}{ccc} F(C) & \xrightarrow{\alpha_C} & G(C) \\ F(f) \downarrow & & \downarrow G(f) \\ F(C') & \xrightarrow{\alpha_{C'}} & G(C') \end{array}$$

Then $\alpha_{C'} \circ F(f) = G(f) \circ \alpha_C$, which implies that $F(f) \circ (\alpha_C)^{-1} = (\alpha_{C'})^{-1} \circ G(f)$, that is, $F(f) \circ \alpha_C^{-1} = \alpha_{C'}^{-1} \circ G(f)$. Hence the first square is commutative, and thus $\alpha^{-1} : G \rightarrow F$ is a natural transformation. For every object C of \mathcal{C} we have $(\alpha^{-1} \circ \alpha)_C = \alpha_C^{-1} \circ \alpha_C = 1_{F(C)}$ and $(\alpha \circ \alpha^{-1})_C = \alpha_C \circ \alpha_C^{-1} = 1_{G(C)}$. Hence $\alpha^{-1} \circ \alpha = 1_F$ and $\alpha \circ \alpha^{-1} = 1_G$. This shows that α is a natural isomorphism. \square

Example 5.5. The twist natural transformation from Example 5.4 is a natural isomorphism.

Example 5.6. Consider the category $\mathbf{Vect}(\mathbb{R})$ of real vector spaces and \mathbb{R} -linear maps. For a real vector space V denote $V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$, which is called the **dual space** of V .

Let $f : V \rightarrow W$ be an \mathbb{R} -linear map. Then it induces a **dual \mathbb{R} -linear map** $f^* : W^* \rightarrow V^*$ defined by $f^*(A) = A \circ f$. Hence we obtain a contravariant functor

$$(-)^* = \text{Hom}_{\mathbb{R}}(-, \mathbb{R}) : \mathbf{Vect}(\mathbb{R})^{\text{op}} \rightarrow \mathbf{Vect}(\mathbb{R}).$$

We define a natural transformation $\eta : 1_{\mathbf{Vect}(\mathbb{R})} \rightarrow (-)^{**}$, where for every $V \in \mathbf{Vect}(\mathbb{R})$, we have $\eta_V : V \rightarrow V^{**}$ defined by $\eta_V(v) = ev_v : V^* \rightarrow \mathbb{R}$ and $ev_v(A) = A(v)$ for every $A \in V^* = \text{Hom}_{\mathbb{R}}(V, \mathbb{R})$.

Let us first show that each η_V is an arrow in the category $\mathbf{Vect}(\mathbb{R})$, that is, an \mathbb{R} -linear map. Let $k_1, k_2 \in \mathbb{R}$ and $v_1, v_2 \in V$. For every $A \in V^*$, we have

$$\begin{aligned} ev_{k_1 v_1 + k_2 v_2}(A) &= A(k_1 v_1 + k_2 v_2) = k_1 A(v_1) + k_2 A(v_2) \\ &= k_1 ev_{v_1}(A) + k_2 ev_{v_2}(A) = [k_1 ev_{v_1} + k_2 ev_{v_2}](A), \end{aligned}$$

whence we have $ev_{k_1 v_1 + k_2 v_2} = k_1 ev_{v_1} + k_2 ev_{v_2}$. It follows that:

$$\eta_V(k_1 v_1 + k_2 v_2) = ev_{k_1 v_1 + k_2 v_2} = k_1 ev_{v_1} + k_2 ev_{v_2} = k_1 \eta_V(v_1) + k_2 \eta_V(v_2).$$

Hence each η_V is an \mathbb{R} -linear map. One may show that η_V is a K -isomorphism if and only if V is finite-dimensional.

Now we check the commutativity of the following diagram:

$$\begin{array}{ccc} V & \xrightarrow{\eta_V} & V^{**} \\ f \downarrow & & \downarrow f^{**} \\ W & \xrightarrow{\eta_W} & W^{**} \end{array}$$

Let $v \in V$. We have $(\eta_W \circ f)(v) = ev_{f(v)} : W^* \rightarrow \mathbb{R}$ and $(f^{**} \circ \eta_V)(v) = ev_v \circ f^* : W^* \rightarrow \mathbb{R}$. For every $A \in W^*$ we have $(ev_v \circ f^*)(A) = ev_v(A \circ f) = (A \circ f)(v) = ev_{f(v)}(A)$, hence $ev_{f(v)} = ev_v \circ f^*$. It follows that $\eta_W \circ f = f^{**} \circ \eta_V$. This shows that η is a natural transformation.

5.3 Equivalence of Categories

Example 5.7. Let \mathbf{Ord}_{fin} be the category of finite ordinal numbers, whose objects are the sets $0, 1, \dots, n$, where $0 = \emptyset$ and $n = \{0, \dots, n-1\}$, and the arrows are the functions between these sets. For each finite set we select an ordinal $|A|$ that is its cardinal and an isomorphism $A \cong |A|$. Then for each function $f : A \rightarrow B$ between finite sets, we have a function $|f|$ by completing the square

$$\begin{array}{ccc} A & \xrightarrow{\cong} & |A| \\ f \downarrow & & \downarrow |f| \\ B & \xrightarrow{\cong} & |B| \end{array}$$

In this way, we obtain a functor $F = |-| : \mathbf{Set}_{fin} \rightarrow \mathbf{Ord}_{fin}$. In fact, all the above maps are in \mathbf{Set}_{fin} . We have the inclusion functor $i : \mathbf{Ord}_{fin} \rightarrow \mathbf{Set}_{fin}$. For every finite set A we have an isomorphism $\nu_A : A \rightarrow (i \circ F)(A) = i(|A|)$ and the above commutative square implies that $i(|f|) \circ \nu_A = \nu_B \circ f$. Hence we have a natural isomorphism

$$\nu : 1_{\mathbf{Set}_{fin}} \rightarrow i \circ F = i \circ |-|$$

between the functors $1_{\mathbf{Set}_{fin}}, i \circ F : \mathbf{Set}_{fin} \rightarrow \mathbf{Set}_{fin}$. Also, we have

$$F \circ i = |i(-)| = 1_{\mathbf{Ord}_{fin}} : \mathbf{Ord}_{fin} \rightarrow \mathbf{Ord}_{fin}$$

This is because, for any finite ordinal n , $|i(n)| = n$, and we can assume that we take $\nu_n = 1_n : n \rightarrow |i(n)|$, so that also $|i(f)| = f : n \rightarrow m$. Note that the two categories are very similar, but they are not the same, and not even isomorphic.

The above example is an equivalence of categories in the following sense.

Definition 5.7. An **isomorphism** of categories consists of a pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ G = 1_{\mathcal{D}}$ and $G \circ F = 1_{\mathcal{C}}$.

An **equivalence** of categories consists of a pair of functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$, and a pair of natural isomorphisms $1_{\mathcal{D}} \cong F \circ G$ and $G \circ F \cong 1_{\mathcal{C}}$. Here F and G are called **pseudo-inverses** and we write $\mathcal{C} \simeq \mathcal{D}$.

A **duality** between categories \mathcal{C} and \mathcal{D} is an equivalence between \mathcal{C} and the opposite category \mathcal{D}^{op} of \mathcal{D} .

Example 5.8. (1) Two poset categories are equivalent if and only if the posets are isomorphic.

(2) Let A be a set and let \mathcal{C} be the poset category of the poset $(\mathcal{P}(A), \subseteq)$, where $\mathcal{P}(A)$ is the power set of A , which consists of all subsets of A . Recall that the only arrows are given by the inclusions. We show that there is an isomorphism of categories between \mathcal{C} and its opposite category \mathcal{C}^{op} . This also shows that there is a duality between \mathcal{C} and itself.

Let $F : \mathcal{C} \rightarrow \mathcal{C}^{op}$ be the functor defined on objects X of \mathcal{C} by $F(X) = A \setminus X$ (set difference) and on arrows (inclusions) by $F(X \rightarrow Y) = (A \setminus X) \rightarrow (A \setminus Y)$. The latter is an arrow in \mathcal{C}^{op} , because $(A \setminus Y) \rightarrow (A \setminus X)$ is an arrow in \mathcal{C} , being the inclusion, because $X \rightarrow Y$ is an inclusion.

Also, let $G : \mathcal{C}^{op} \rightarrow \mathcal{C}$ be the functor defined on objects by $G(X) = A \setminus X$ and on arrows by $G(Y \rightarrow X) = (A \setminus Y) \rightarrow (A \setminus X)$. The latter is an arrow in \mathcal{C} , being inclusion, because $X \rightarrow Y$ is an inclusion as $Y \rightarrow X$ is an arrow in \mathcal{C}^{op} .

It is easy to see that $(G \circ F)(X) = X$ for every object X of \mathcal{C} and $(G \circ F)(X \rightarrow Y) = X \rightarrow Y$ for every arrow (inclusion) in \mathcal{C} . Hence $G \circ F = 1_{\mathcal{C}}$. Also, one shows that $F \circ G = 1_{\mathcal{C}^{op}}$. Therefore, the categories \mathcal{C} and \mathcal{C}^{op} are isomorphic.

Remark 5.6. (1) The equivalence of categories is the more relevant out of the two concepts, the isomorphism of categories being too restrictive.

(2) Every isomorphism of categories is an equivalence of categories, but not conversely. In Example 5.7 we have seen that we have an equivalence of categories $\mathbf{Ord}_{fin} \simeq \mathbf{Set}_{fin}$ given by the above functors F and i . But $i \circ F \neq 1_{\mathbf{Set}_{fin}}$, hence the functors F and i do not define an isomorphism of categories.

A useful practical characterization of equivalences of categories is the following one.

Theorem 5.7. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then the following are equivalent:

- (1) F is (part of) an equivalence of categories.
- (2) F is fully faithful, essentially injective on objects and essentially surjective on objects.
- (3) F is fully faithful and essentially surjective on objects.

Proof. (1) \implies (2) Suppose that F is an equivalence of categories. Then there is a functor $E : \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms $\alpha : 1_{\mathcal{D}} \rightarrow F \circ E$ and $\beta : E \circ F \rightarrow 1_{\mathcal{C}}$. Then for every object C of \mathcal{C} , $\beta_C : (E \circ F)(C) \rightarrow C$ is an isomorphism, and for every arrow $f : C \rightarrow C'$ in \mathcal{C} we have the following commutative diagram:

$$\begin{array}{ccc} (E \circ F)(C) & \xrightarrow{\beta_C} & C \\ (E \circ F)(f) \downarrow & & \downarrow f \\ (E \circ F)(C') & \xrightarrow{\beta_{C'}} & C' \end{array}$$

We show that F is faithful. Let f, g be arrows in \mathcal{C} such that $F(f) = F(g)$. Then $(E \circ F)(f) = (E \circ F)(g)$, whence $\beta_{C'} \circ (E \circ F)(f) = \beta_{C'} \circ (E \circ F)(g)$. Using the above commutative square for f and g , we have $f \circ \beta_C = \beta_{C'} \circ (E \circ F)(f) = \beta_{C'} \circ (E \circ F)(g) = g \circ \beta_C$, whence $f = g$, because β_C is an isomorphism. Hence F is faithful.

We show that F is full. Let $h : F(C) \rightarrow F(C')$ be an arrow in \mathcal{D} . We have

$$C \xrightarrow{\beta_C^{-1}} (E \circ F)(C) \xrightarrow{E(h)} (E \circ F)(C') \xrightarrow{\beta_{C'}} C'.$$

Denote $f = \beta_{C'} \circ E(h) \circ \beta_C^{-1}$. Then $f \circ \beta_C = \beta_{C'} \circ E(h)$, hence the diagram

$$\begin{array}{ccc} (E \circ F)(C) & \xrightarrow{\beta_C} & C \\ E(h) \downarrow & & \downarrow f \\ (E \circ F)(C') & \xrightarrow{\beta_{C'}} & C' \end{array}$$

is commutative. By this diagram and the first diagram, we must have $E(h) = (E \circ F)(f)$. Since E is faithful, we deduce that $h = F(f)$. Hence F is full.

Note that F is essentially injective on objects by Lemma 5.1.

Finally, for every object D of \mathcal{D} we have $D \cong (F \circ E)(D) = F(E(D))$ by using the isomorphism α_D . Hence F is essentially surjective on objects.

(2) \implies (3) This is clear.

(3) \implies (1) Suppose that F is fully faithful and essentially surjective on objects. Then for every object D of \mathcal{D} we may choose an object C of \mathcal{C} (unique up to isomorphism by Lemma 5.1) such that $D \cong F(C)$. Hence to every object D of \mathcal{D} we may associate an object $E(D) = C$ (unique up to isomorphism) and an arrow $\alpha_D : D \rightarrow F(C) = (F \circ E)(D)$ which is an isomorphism.

Let $g : D \rightarrow D'$ be an arrow in \mathcal{D} . Consider the arrow $\alpha_{D'} \circ g \circ \alpha_D^{-1} : (F \circ E)(D) \rightarrow (F \circ E)(D')$. Since F is fully faithful, there is a unique arrow $f : E(D) \rightarrow E(D')$ such that $\alpha_{D'} \circ g \circ \alpha_D^{-1} = F(f)$. Then we define $E(g) = f$. Let us show that this assignment yields a functor. Let $g : D \rightarrow D'$ and $g' : D' \rightarrow D''$ be arrows in \mathcal{D} . Then there are unique arrows $f : E(D) \rightarrow E(D')$ and $f' : E(D') \rightarrow E(D'')$ such that $\alpha_{D'} \circ g \circ \alpha_D^{-1} = F(f)$ and $\alpha_{D''} \circ g' \circ \alpha_{D'}^{-1} = F(f')$. It follows that:

$$\begin{aligned} \alpha_{D''} \circ (g' \circ g) \circ \alpha_D^{-1} &= (\alpha_{D''} \circ g' \circ \alpha_{D'}^{-1}) \circ (\alpha_{D'} \circ g \circ \alpha_D^{-1}) \\ &= F(f') \circ F(f) \\ &= F(f' \circ f). \end{aligned}$$

Thus the arrow associated to $g' \circ g$ is $f' \circ f$, that is, $E(g' \circ g) = f' \circ f$. Then we have $F(E(g' \circ g)) = F(f' \circ f) = F(E(g') \circ E(g))$, which implies $E(g' \circ g) = E(g') \circ E(g)$, because F is faithful. Also, $(F \circ E)(1_D) = \alpha_D \circ 1_D \circ \alpha_D^{-1} = 1_{(F \circ E)(D)} = F(1_{E(D)})$. This implies that $E(1_D) = 1_{E(D)}$, because F is faithful. Hence $E : \mathcal{D} \rightarrow \mathcal{C}$ is a functor.

We have $(F \circ E)(g) \circ \alpha_D = F(f) \circ \alpha_D = \alpha_{D'} \circ g \circ \alpha_D^{-1} \circ \alpha_D = \alpha_{D'} \circ g$, hence the following diagram is commutative:

$$\begin{array}{ccc} D & \xrightarrow{\alpha_D} & (F \circ E)(D) \\ g \downarrow & & \downarrow (F \circ E)(g) \\ D' & \xrightarrow{\alpha_{D'}} & (F \circ E)(D') \end{array}$$

This shows that we have a natural transformation $\alpha : 1_{\mathcal{D}} \rightarrow F \circ E$, which is a natural isomorphism, because each component α_D is an isomorphism.

In order to construct a natural transformation $\beta : E \circ F \rightarrow 1_{\mathcal{C}}$, for every object C of \mathcal{C} consider the isomorphism $\alpha_{F(C)} : F(C) \rightarrow (F \circ E)(F(C))$. Then $\alpha_{F(C)}^{-1} : (F \circ E)(F(C)) \rightarrow F(C)$. Since F

is fully faithful, there is a unique arrow $\beta_C : (E \circ F)(C) \rightarrow C$ such that $\alpha_{F(C)}^{-1} = F(\beta_C)$, and a unique arrow $\gamma_C : C \rightarrow (E \circ F)(C)$ such that $\alpha_{F(C)} = F(\gamma_C)$. We have

$$F(\beta_C \circ \gamma_C) = F(\beta_C) \circ F(\gamma_C) = \alpha_{F(C)}^{-1} \circ \alpha_{F(C)} = 1_{(F \circ E \circ F)(C)} = F(1_{(E \circ F)(C)}),$$

which implies that $\beta_C \circ \gamma_C = 1_C$, because F is faithful. Similarly, one has $\gamma_C \circ \beta_C = 1_{(E \circ F)(C)}$. Hence each β_C is an isomorphism.

We show that β is a natural transformation. Let $f : C \rightarrow C'$ be an arrow in \mathcal{C} . Since $\alpha^{-1} : F \circ E \rightarrow 1_{\mathcal{D}}$ is a natural transformation, we have the following commutative diagram:

$$\begin{array}{ccc} (F \circ E \circ F)(C) & \xrightarrow{\alpha_{F(C)}^{-1}} & F(C) \\ (F \circ E \circ F)(f) \downarrow & & \downarrow F(f) \\ (F \circ E \circ F)(C') & \xrightarrow{\alpha_{F(C')}^{-1}} & F(C') \end{array}$$

It follows that $F(f \circ \beta_C) = F(f) \circ \alpha_{F(C)}^{-1} = \alpha_{F(C')}^{-1} \circ (F \circ E \circ F)(f) = F(\beta_{C'} \circ (E \circ F)(f))$. Since F is faithful, we have $f \circ \beta_C = \beta_{C'} \circ (E \circ F)(f)$. Then the following diagram is commutative:

$$\begin{array}{ccc} (E \circ F)(C) & \xrightarrow{\beta_C} & C \\ (E \circ F)(f) \downarrow & & \downarrow f \\ (E \circ F)(C') & \xrightarrow{\beta_{C'}} & C' \end{array}$$

Hence $\beta : E \circ F \rightarrow 1_{\mathcal{C}}$ is a natural transformation, which is a natural isomorphism, because each component β_C is an isomorphism.

Hence F is an equivalence of categories. \square

We have a similar result for isomorphisms of categories.

Theorem 5.8. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor. Then the following are equivalent:

- (1) F is (part of) an isomorphism of categories.
- (2) F is fully faithful, injective on objects and surjective on objects.

Proof. We give a direct proof, which is a simplification of the proof for equivalence of categories.

(1) \implies (2) Suppose that F is an isomorphism of categories. Then there is a functor $E : \mathcal{D} \rightarrow \mathcal{C}$ such that $F \circ E = 1_{\mathcal{D}}$ and $E \circ F = 1_{\mathcal{C}}$. If f, g are arrows in \mathcal{C} such that $F(f) = F(g)$, then $(E \circ F)(f) = (E \circ F)(g)$, whence $f = g$. Hence F is faithful. If $h : F(C) \rightarrow F(C')$ is an arrow in \mathcal{D} , then $h = F(E(h))$. Hence F is full. If A, B are objects of \mathcal{C} such that $F(A) = F(B)$, then $(E \circ F)(A) = (E \circ F)(B)$, whence $A = B$. Hence F is injective on objects. If D is an object of \mathcal{D} , then $D = (F \circ E)(D) = F(E(D))$. Hence F is surjective on objects.

(2) \implies (1) Suppose that F is fully faithful, injective on objects and surjective on objects. Then for every object D of \mathcal{D} we have an object C of \mathcal{C} such that $D = F(C)$. If C' is another object of \mathcal{C} such that $D = F(C')$, then we have $F(C) = F(C')$, which implies that $C = C'$, because F is injective on objects. Hence to every object D of \mathcal{D} we may associate a unique object $E(D) = C$. Now let $g : D \rightarrow D'$ be an arrow in \mathcal{D} . Then $(F \circ E)(g) : (F \circ E)(D) \rightarrow (F \circ E)(D')$. Since F is fully faithful, there is a unique arrow $f : E(D) \rightarrow E(D')$ such that $g = F(f)$. Then we define $E(g) = f$. Let us show that E defines a functor. Let $g : D \rightarrow D'$ and $g' : D' \rightarrow D''$ be arrows in \mathcal{D} . Then there are unique arrows $f : E(D) \rightarrow E(D')$ and $f' : E(D') \rightarrow E(D'')$ such that $g = F(f)$ and $g' = F(f')$. It follows that $g' \circ g = F(f') \circ F(f) = F(f' \circ f)$. Thus the arrow associated to $g' \circ g$ is $f' \circ f$, that is, $E(g' \circ g) = f' \circ f$. Then we have $F(E(g' \circ g)) = F(f' \circ f) = F(E(g') \circ E(g))$, which implies $E(g' \circ g) = E(g') \circ E(g)$, because F is faithful. Also, $(F \circ E)(1_D) = 1_D = 1_{(F \circ E)(D)} = F(1_{E(D)})$. This implies that $E(1_D) = 1_{E(D)}$, because F is faithful. Hence $E : \mathcal{D} \rightarrow \mathcal{C}$ is a functor.

Finally, we have $(F \circ E)(g) = F(E(g)) = F(f) = g$ for every arrow g in \mathcal{D} , and $(E \circ F)(f) = E(F(f)) = f$ for every arrow f in \mathcal{C} . Hence F is an isomorphism of categories. \square

Example 5.9. (1) Let \mathcal{C} be a category with two objects C_1 and C_2 , and two non-trivial morphisms $f : C_1 \rightarrow C_2$ and $g : C_2 \rightarrow C_1$ such that $g \circ f = 1_{C_1}$ and $f \circ g = 1_{C_2}$, and let \mathcal{D} be a category with one object D and one morphism 1_D . Then the functor $F : \mathcal{C} \rightarrow \mathcal{D}$ given by $F(C_1) = F(C_2) = D$ and $F(f) = 1_D$ is fully faithful and essentially surjective on objects. Hence it is an equivalence of categories, but it is not an isomorphism of categories.

(2) Let $U : \mathbf{Vect}(K) \rightarrow \mathbf{Ab}$ the forgetful functor. Let us show that U is a faithful functor, which is essentially surjective on objects, but it is not full.

Recall that a K -linear map between two K -vector spaces is a group homomorphism between the underlying abelian groups which respects the scalar multiplication. So two distinct K -linear maps should already be distinct as group homomorphisms of underlying groups and respect the scalar multiplication. Hence U is faithful.

Given the field K and an abelian group G , one may define the trivial K -vector space ${}_K G$ on G by taking $k \cdot g = g$ for every $g \in G$. Then $U({}_K G) = G$, which shows that U is essentially surjective on objects.

Note that the endomorphism ring of the canonical K -vector space K is isomorphic to K . But in general the endomorphism ring of the underlying abelian group $(K, +)$ is not isomorphic to $(K, +, \cdot)$ (e.g., take $K = \mathbb{C}$). Hence U is not full.

Example 5.10. Let $\mathbf{FDVect}(K)$ be the category of non-zero finite-dimensional K -vector spaces and K -linear maps. Note that $\mathbf{FDVect}(K)$ is a full subcategory of the category $\mathbf{Vect}(K)$. Consider also the category $\mathbf{Mat}(K)$ defined as follows:

- objects: the non-zero natural numbers
- arrows: the matrices in $M_{m,n}(K)$ for $m, n \in \mathbb{N}^*$
- composition: the multiplication of matrices
- identity arrow: the identity matrix

We show that there is an equivalence of categories between $\mathbf{FDVect}(K)$ and $\mathbf{Mat}(K)$.

Let $F : \mathbf{FDVect}(K) \rightarrow \mathbf{Mat}(K)$ be defined on objects V of $\mathbf{FDVect}(K)$ by $F(V) = \dim_K(V)$, and on arrows $f : V \rightarrow V'$ by $F(f) = [f]_{BB'}$, where B and B' are some fixed bases of V and V' respectively. In this way we obtain a functor F .

We show that F is faithful. Let $f, g : V \rightarrow V'$ be arrows in $\mathbf{FDVect}(K)$, that is, K -linear maps, such that $F(f) = F(g)$. Then $[f]_{BB'} = [g]_{BB'}$. But K -linear maps are determined by their matrices, hence $f = g$. Thus, F is faithful.

We show that F is full. Let $A \in M_{m,n}(K)$, that is, an arrow in $\mathbf{Mat}(K)$ for some $m, n \in \mathbb{N}^*$. Consider a K -linear map $f : K^n \rightarrow K^m$ such that $[f]_{BB'} = A$. Then $F(f) = A$. Thus, F is full.

We show that F is essentially surjective on objects. Let $n \in \mathbb{N}^*$, that is, an object in $\mathbf{Mat}(K)$. Consider the object $V = K^n$ of $\mathbf{FDVect}(K)$. Then $n = \dim_K(V) = F(V)$. Thus, F is essentially surjective on objects.

Therefore, F is an equivalence of categories by Theorem 5.8. Note that F is not an isomorphism of categories.

Chapter 6

Categories of Diagrams

6.1 The Yoneda Lemma

Throughout this section all categories are considered to be locally small.

Definition 6.1. The covariant Yoneda functor is the functor

$$y : \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set}),$$

where $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ is the category of contravariant functors from \mathcal{C} to \mathbf{Set} (also called **presheaves**), defined on objects C of \mathcal{C} by (the contravariant functor)

$$y(C) = \text{Hom}_{\mathcal{C}}(-, C) : \mathcal{C} \rightarrow \mathbf{Set}$$

and on arrows $f : C \rightarrow D$ in \mathcal{C} by (the natural transformation)

$$y(f) = \text{Hom}_{\mathcal{C}}(-, f) : y(C) = \text{Hom}_{\mathcal{C}}(-, C) \rightarrow y(D) = \text{Hom}_{\mathcal{C}}(-, D).$$

Remark 6.1. Let us give the explicit definition of the covariant Yoneda functor, which will be useful for the proof and the consequences of the Yoneda lemma.

Let C be an object of \mathcal{C} and let $f : C \rightarrow D$ be an arrow in \mathcal{C} .

- The contravariant functor

$$y(C) = \text{Hom}_{\mathcal{C}}(-, C) : \mathcal{C} \rightarrow \mathbf{Set}$$

is defined on objects C' of \mathcal{C} by

$$(y(C))(C') = \text{Hom}_{\mathcal{C}}(C', C)$$

and on arrows $g : C' \rightarrow C''$ by

$$(y(C))(g) = \text{Hom}_{\mathcal{C}}(g, C) : \text{Hom}_{\mathcal{C}}(C'', C) \rightarrow \text{Hom}_{\mathcal{C}}(C', C),$$

where for every $h \in \text{Hom}_{\mathcal{C}}(C'', C)$ we have:

$$[(y(C))(g)](h) = [\text{Hom}_{\mathcal{C}}(g, C)](h) = h \circ g.$$

- The component of the natural transformation

$$y(f) = \text{Hom}_{\mathcal{C}}(-, f) : \text{Hom}_{\mathcal{C}}(-, C) \rightarrow \text{Hom}_{\mathcal{C}}(-, D)$$

at an object C' of \mathcal{C} is defined by

$$y(f)_{C'} = \text{Hom}_{\mathcal{C}}(C', f) : \text{Hom}_{\mathcal{C}}(C', C) \rightarrow \text{Hom}_{\mathcal{C}}(C', D),$$

where for every $h \in \text{Hom}_{\mathcal{C}}(C', C)$ we have:

$$[y(f)_{C'}](h) = [\text{Hom}_{\mathcal{C}}(C', f)](h) = f \circ h.$$

Lemma 6.2 (Yoneda). Let \mathcal{C} be a locally small category. Then for every object C of \mathcal{C} and for every object F of $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$, there is an isomorphism in \mathbf{Set} :

$$\text{Hom}_{\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})}(y(C), F) \cong F(C),$$

which is natural in both C and F .

Proof. For the sake of simplicity, we omit to write the index of $\text{Hom}_{\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})}(y(C), F)$.

Step 1: Define two natural transformations.

Let C be an object of \mathcal{C} and let F be an object of $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$.

We define a natural transformation $\eta_{C,F} : \text{Hom}(y(C), F) \rightarrow F(C)$ as follows. For every $\nu \in \text{Hom}(y(C), F)$ (natural transformation), take

$$\eta_{C,F}(\nu) = \nu_C(1_C),$$

where $\nu_C : (y(C))(C) = \text{Hom}_{\mathcal{C}}(C, C) \rightarrow F(C)$. Note that $\nu_C(1_C) \in F(C)$.

Also, we define a natural transformation $\mu_{C,F} : F(C) \rightarrow \text{Hom}(y(C), F)$ as follows. For every $a \in F(C)$, let $\nu_a : y(C) \rightarrow F$ be given by

$$(\nu_a)_{C'} : (y(C))(C') = \text{Hom}_{\mathcal{C}}(C', C) \rightarrow F(C')$$

where $(\nu_a)_{C'}(h) = [F(h)](a)$ for every $h \in \text{Hom}_{\mathcal{C}}(C', C)$. Note that $F(h) : F(C) \rightarrow F(C')$. Then we take $\mu_{C,F}(a) = \nu_a$.

We show that $\nu_a : y(C) \rightarrow F$ is a natural transformation. Let $f : C'' \rightarrow C'$ be an arrow in \mathcal{C} . Consider the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathcal{C}}(C', C) & \xrightarrow{(\nu_a)_{C'}} & F(C') \\ (y(C))(f) = \text{Hom}_{\mathcal{C}}(f, C) \downarrow & & \downarrow F(f) \\ \text{Hom}_{\mathcal{C}}(C'', C) & \xrightarrow{(\nu_a)_{C''}} & F(C'') \end{array}$$

For every $h \in \text{Hom}_{\mathcal{C}}(C', C)$, we have:

$$\begin{aligned} [(\nu_a)_{C''} \circ \text{Hom}_{\mathcal{C}}(f, C)](h) &= (\nu_a)_{C''}(\text{Hom}_{\mathcal{C}}(f, C)(h)) = (\nu_a)_{C''}(h \circ f) \\ &= [F(h \circ f)](a) = (F(f) \circ F(h))(a) = F(f)[(F(h))(a)] \\ &= (F(f))[(\nu_a)_{C'}(h)] = [F(f) \circ (\nu_a)_{C'}](h), \end{aligned}$$

hence the above diagram is commutative. Therefore, ν_a is a natural transformation, and thus $\nu_a \in \text{Hom}(y(C), F)$.

Step 2: Show that their components are isomorphisms.

We show that $\mu_{C,F} \circ \eta_{C,F} = 1_{\text{Hom}(y(C), F)}$. Let $\nu \in \text{Hom}(y(C), F)$ and $h \in \text{Hom}_{\mathcal{C}}(C', C)$. We have:

$$(\mu_{C,F} \circ \eta_{C,F})(\nu) = \mu_{C,F}(\nu_C(1_C)) = \nu_{\nu_C(1_C)}.$$

We need to prove that $\nu_{\nu_C(1_C)} = \nu$. Since ν is a natural transformation, the following diagram is commutative:

$$\begin{array}{ccc} (y(C))(C) & \xrightarrow{\nu_C} & F(C) \\ (y(C))(h) \downarrow & & \downarrow F(h) \\ (y(C))(C') & \xrightarrow{\nu_{C'}} & F(C') \end{array}$$

hence $F(h) \circ \nu_C = \nu_{C'} \circ [y(C)](h)$. By using also Remark 6.1, it follows that:

$$\begin{aligned} (\nu_{\nu_C(1_C)})_{C'}(h) &= [F(h)](\nu_C(1_C)) = (F(h) \circ \nu_C)(1_C) = (\nu_{C'} \circ [y(C)](h))(1_C) \\ &= \nu_{C'}([(y(C))(h)](1_C)) = \nu_{C'}(1_C \circ h) = \nu_{C'}(h). \end{aligned}$$

This shows that $\nu_{\nu_C(1_C)} = \nu$. It follows that $\mu_{C,F} \circ \eta_{C,F} = 1_{\text{Hom}(y(C), F)}$.

We show that $\eta_{C,F} \circ \mu_{C,F} = 1_{F(C)}$. Let $a \in F(C)$. We have:

$$\begin{aligned} (\eta_{C,F} \circ \mu_{C,F})(a) &= \eta_{C,F}(\mu_{C,F}(a)) = \eta_{C,F}(\nu_a) \\ &= (\nu_a)_C(1_C) = [F(1_C)](a) = 1_{F(C)}(a) = a. \end{aligned}$$

This shows that $\eta_{C,F} \circ \mu_{C,F} = 1_{F(C)}$. Hence $\eta_{C,F}$ and $\mu_{C,F}$ are inverse to each other.

Step 3: Show naturality in C .

Let $f : C' \rightarrow C$ be an arrow in \mathcal{C} and let F be an object of $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$. We show that the following diagram is commutative:

$$\begin{array}{ccc} \text{Hom}(y(C), F) & \xrightarrow{\eta_{C,F}} & F(C) \\ \text{Hom}(y(f), F) \downarrow & & \downarrow F(f) \\ \text{Hom}(y(C'), F) & \xrightarrow{\eta_{C',F}} & F(C') \end{array}$$

hence $F(f) \circ \nu_C = \nu_{C'} \circ (y(C))(f)$. By using also Remark 6.1, for every $\nu \in \text{Hom}(y(C), F)$, we have:

$$\begin{aligned} (\eta_{C',F} \circ \text{Hom}(y(f), F))(\nu) &= \eta_{C',F}[\text{Hom}(y(f), F)(\nu)] = \eta_{C',F}(\nu \circ y(f)) \\ &= (\nu \circ y(f))_{C'}(1_{C'}) = (\nu_{C'} \circ y(f)_{C'})(1_{C'}) \\ &= \nu_{C'}(y(f)_{C'}(1_{C'})) = \nu_{C'}(1_{C'} \circ f) = \nu_{C'}(f) \\ &= \nu_{C'}(f \circ 1_C) = \nu_{C'}[(y(C))(f)(1_C)] \\ &= [\nu_{C'} \circ (y(C))(f)](1_C) = (F(f) \circ \nu_C)(1_C) \\ &= F(f)(\nu_C(1_C)) = F(f)(\eta_{C,F}(\nu)) \\ &= (F(f) \circ \eta_{C,F})(\nu). \end{aligned}$$

This shows that the above diagram is commutative, and thus we have naturality in C .

Step 4: Show naturality in F .

Let $\Phi : F \rightarrow F'$ be an arrow in $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ and let C be an object of \mathcal{C} . We show that the following diagram is commutative:

$$\begin{array}{ccc} \text{Hom}(y(C), F) & \xrightarrow{\eta_{C,F}} & F(C) \\ \text{Hom}(y(f), \Phi) \downarrow & & \downarrow \Phi_C \\ \text{Hom}(y(C), F') & \xrightarrow{\eta_{C,F'}} & F'(C) \end{array}$$

By using also Remark 6.1, for every $\nu \in \text{Hom}(y(C), F)$, we have

$$\begin{aligned} (\eta_{C,F'} \circ \text{Hom}(y(C), \Phi))(\nu) &= \eta_{C,F'}[\text{Hom}(y(C), \Phi)(\nu)] = \eta_{C,F'}(\Phi \circ \nu) \\ &= (\Phi \circ \nu)_C(1_C) = (\Phi_C \circ \nu_C)(1_C) \\ &= \Phi_C(\nu_C(1_C)) = \Phi_C(\eta_{C,F}(\nu)) = (\Phi_C \circ \eta_{C,F})(\nu) \end{aligned}$$

This shows that the above diagram is commutative, and thus we have naturality in F . \square

Definition 6.2. A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called an **embedding** if F is fully faithful and injective on objects in the sense that if A, B are objects of \mathcal{C} such that $F(A) = F(B)$, then $A = B$.

Corollary 6.3. The covariant Yoneda functor is an embedding.

Proof. We show that y is fully faithful. Let C and D be objects of \mathcal{C} . By using the Yoneda lemma with $F = y(D)$ and Remark 6.1 we have:

$$\text{Hom}_{\mathcal{C}}(C, D) \cong (y(D))(C) \cong \text{Hom}_{\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})}(y(C), y(D)).$$

We need to show that this isomorphism is induced by y . For every $h \in (y(D))(C) = \text{Hom}_{\mathcal{C}}(C, D)$, the isomorphism yields a natural transformation $\nu_h : y(C) \rightarrow y(D)$ defined as follows. For every arrow $f : C' \rightarrow C$ in \mathcal{C} , we have:

$$(\nu_h)_{C'}(f) = [y(D)(f)](h) = h \circ f = (y(h))_{C'}(f),$$

where $y(h) : y(C) \rightarrow y(D)$ and

$$(y(h))_{C'} = \text{Hom}_{\mathcal{C}}(C', h) : (y(C))(C') = \text{Hom}_{\mathcal{C}}(C', C) \rightarrow (y(D))(C') = \text{Hom}_{\mathcal{C}}(C', D).$$

Hence $\nu_h = y(h)$, which shows that the above isomorphism is induced by y . Therefore, y is fully faithful.

We show that y is injective on objects. Let C and D be objects of \mathcal{C} such that $y(C) = y(D)$. We have:

$$1_C \in \text{Hom}_{\mathcal{C}}(C, C) = (y(C))(C) = (y(D))(C) = \text{Hom}_{\mathcal{C}}(C, D),$$

which implies that $C = D$. Hence y is injective on objects.

Therefore, y is an embedding. \square

Remark 6.4.

- (1) If \mathcal{C} is small, then $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ is locally small, and thus $\text{Hom}(y(C), F) \in \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ is a set.
- (2) If \mathcal{C} is locally small, then $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ need not be locally small. In this case, the Yoneda lemma tells us that $\text{Hom}(y(C), F) \in \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ is always a set.
- (3) If \mathcal{C} is not locally small, then $y : \mathcal{C} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ will not even be defined, and thus the Yoneda lemma does not apply.
- (4) Taking $F = y(D)$ in the Yoneda lemma, we have:

$$\text{Hom}(\text{Hom}_{\mathcal{C}}(-, C), \text{Hom}_{\mathcal{C}}(-, D)) \cong \text{Hom}(y(C), y(D)) \cong (y(D))(C) = \text{Hom}_{\mathcal{C}}(C, D).$$

Dually, there is a contravariant Yoneda functor and the corresponding Yoneda lemma.

Definition 6.3. The contravariant Yoneda functor is the functor

$$y : \mathcal{C}^{\text{op}} \rightarrow \text{Fun}(\mathcal{C}, \mathbf{Set}),$$

where $\text{Fun}(\mathcal{C}, \mathbf{Set})$ is the category of covariant functors from \mathcal{C} to \mathbf{Set} , defined on objects C of \mathcal{C} by (the covariant functor)

$$y(C) = \text{Hom}_{\mathcal{C}}(C, -) : \mathcal{C} \rightarrow \mathbf{Set}$$

and on arrows $f : C \rightarrow D$ in \mathcal{C} by (the natural transformation)

$$y(f) = \text{Hom}_{\mathcal{C}}(f, -) : y(D) = \text{Hom}_{\mathcal{C}}(D, -) \rightarrow y(C) = \text{Hom}_{\mathcal{C}}(C, -).$$

Lemma 6.5 (Yoneda). Let \mathcal{C} be a locally small category. Then for every object C of \mathcal{C} and for every object F of $\text{Fun}(\mathcal{C}, \mathbf{Set})$, there is an isomorphism in \mathbf{Set} :

$$\text{Hom}_{\text{Fun}(\mathcal{C}, \mathbf{Set})}(y(C), F) \cong F(C),$$

which is natural in both C and F . In particular, we have:

$$\text{Hom}(\text{Hom}_{\mathcal{C}}(C, -), \text{Hom}_{\mathcal{C}}(D, -)) \cong \text{Hom}(y(C), y(D)) \cong (y(D))(C) = \text{Hom}_{\mathcal{C}}(D, C).$$

Corollary 6.6. The contravariant Yoneda functor is an embedding.

6.2 Applications of the Yoneda Lemma

We have seen that every fully faithful functor is “essentially injective” on objects. Hence we have the following consequence.

Corollary 6.7 (Yoneda principle). Let \mathcal{C} be a locally small category, and let A, B be objects of \mathcal{C} such that $y(A) \cong y(B)$. Then $A \cong B$.

Example 6.1. Let \mathcal{C} be a locally small category with binary products, binary coproducts and exponentials, in the sense that any two objects B and C of \mathcal{C} have an exponential C^B (e.g., \mathbf{Set}). Then it is known that there is an isomorphism (bijection)

$$\mathrm{Hom}_{\mathbf{Set}}(A \times B, C) \cong \mathrm{Hom}_{\mathbf{Set}}(A, C^B),$$

which is natural in A , B and C . Let us show that:

$$(A \oplus B) \times C \cong (A \times C) \oplus (B \times C).$$

One option is to get the isomorphism by using universal mapping property of the product and the coproduct. Alternatively, one may use the Yoneda principle as follows. We first show that for every object X of \mathcal{C} , we have the natural isomorphism:

$$\mathrm{Hom}_{\mathbf{Set}}((A \oplus B) \times C, X) \cong \mathrm{Hom}_{\mathbf{Set}}((A \times C) \oplus (B \times C), X).$$

To this end, note that we have the natural isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{Set}}((A \oplus B) \times C, X) &\cong \mathrm{Hom}_{\mathbf{Set}}(A \oplus B, X^C) \\ &\cong \mathrm{Hom}_{\mathbf{Set}}(A, X^C) \oplus \mathrm{Hom}_{\mathbf{Set}}(B, X^C) \\ &\cong \mathrm{Hom}_{\mathbf{Set}}(A \times C, X) \oplus \mathrm{Hom}_{\mathbf{Set}}(B \times C, X) \\ &\cong \mathrm{Hom}_{\mathbf{Set}}((A \times C) \oplus (B \times C), X). \end{aligned}$$

Now the Yoneda principle implies that $(A \oplus B) \times C \cong (A \times C) \oplus (B \times C)$.

Example 6.2 (Yoneda meets Cayley). Recall that Cayley Theorem states that every group is isomorphic to a subgroup of a symmetric group. Let (G, \cdot) be a group and consider the symmetric group

$$S_G = \{g : G \rightarrow G \mid g \text{ is bijective}\}.$$

For every $a \in G$, define

$$t_a : G \rightarrow G \text{ by } t_a(x) = ax, \quad \forall x \in G.$$

One proves that $t_a \in S_G$, that is, t_a is bijective. We may now define

$$f : G \rightarrow S_G \text{ by } f(a) = t_a, \forall a \in G.$$

One shows that f is an injective group homomorphism. By the first isomorphism theorem, it follows that G is isomorphic to the subgroup $\mathrm{Im} f$ of S_G .

Recall that every monoid (M, \cdot) has an associated category, which has only one object M and has as arrows the elements of M . In particular, we may consider the associated category \mathcal{C} of a group (G, \cdot) . Now consider the Yoneda embedding $y : \mathcal{C} \rightarrow \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$ defined on objects C of \mathcal{C} by (the contravariant functor)

$$y(C) = \mathrm{Hom}_{\mathcal{C}}(-, C) : \mathcal{C} \rightarrow \mathbf{Set}$$

and on arrows $f : C \rightarrow D$ in \mathcal{C} by (the natural transformation)

$$y(f) = \mathrm{Hom}_{\mathcal{C}}(-, f) : y(C) = \mathrm{Hom}_{\mathcal{C}}(-, C) \rightarrow y(D) = \mathrm{Hom}_{\mathcal{C}}(-, D).$$

But the category \mathcal{C} has only one object, namely (G, \cdot) . Let a be an arrow in \mathcal{C} , that is, $a \in \mathrm{Hom}_{\mathcal{C}}(G, G) = G$.

- The contravariant functor $y(G) = \mathrm{Hom}_{\mathcal{C}}(-, G) : \mathcal{C} \rightarrow \mathbf{Set}$ is defined on the object G of \mathcal{C} by

$$(y(G))(G) = \mathrm{Hom}_{\mathcal{C}}(G, G) = G$$

and on arrows $a \in \mathrm{Hom}_{\mathcal{C}}(G, G) = G$ by

$$(y(G))(a) = \mathrm{Hom}_{\mathcal{C}}(a, G) : \mathrm{Hom}_{\mathcal{C}}(G, G) = G \rightarrow \mathrm{Hom}_{\mathcal{C}}(G, G) = G,$$

where for every $x \in \mathrm{Hom}_{\mathcal{C}}(G, G) = G$ we have:

$$[(y(G))(a)](x) = [\mathrm{Hom}_{\mathcal{C}}(a, G)](x) = x \cdot a.$$

- The component of the natural transformation

$$y(a) = \text{Hom}_{\mathcal{C}}(-, a) : \text{Hom}_{\mathcal{C}}(-, G) \rightarrow \text{Hom}_{\mathcal{C}}(-, G)$$

at the object G of \mathcal{C} is defined by

$$y(a)_G = \text{Hom}_{\mathcal{C}}(G, a) : \text{Hom}_{\mathcal{C}}(G, G) = G \rightarrow \text{Hom}_{\mathcal{C}}(G, G) = G,$$

where for every $x \in \text{Hom}_{\mathcal{C}}(G, G) = G$ we have:

$$[y(a)_G](x) = [\text{Hom}_{\mathcal{C}}(G, a)](x) = a \cdot x,$$

Note that $y(a)_G = t_a$.

By the Yoneda lemma and the Yoneda embedding, we have the following (natural) isomorphism f :

$$G = \text{Hom}_{\mathcal{C}}(G, G) \cong \text{Hom}(\text{Hom}_{\mathcal{C}}(-, G), \text{Hom}_{\mathcal{C}}(-, G)),$$

which is given by $f(a) = y(a)$. The natural transformation $y(a)$ has only one component, namely $y(a)_G = t_a$.

Example 6.3 (Yoneda meets Linear Algebra). Let K be a field. Consider the category $\mathbf{Mat}(K)$, denoted simply by \mathbf{Mat} , defined as follows:

- objects: the non-zero natural numbers
- arrows: the matrices in $M_{m,n}(K)$ for $m, n \in \mathbb{N}^*$
- composition: the multiplication of matrices
- identity arrow: the identity matrix

For every $k \in \mathbb{N}^*$, the set of all matrices with k columns is organized by the data of the k -column functor $h_k : \mathbf{Mat} \rightarrow \mathbf{Set}$ defined on objects $n \in \mathbb{N}^*$ by the set

$$h_k(n) = M_{n,k}(K)$$

and on arrows $A : m \rightarrow n$ in \mathbf{Mat} , that is, $A \in M_{m,n}(K)$ by the function given by the left multiplication by A :

$$h_k(A) : h_k(n) = M_{n,k}(K) \rightarrow h_k(m) = M_{m,k}(K), \quad [h_k(A)](C) = A \cdot C.$$

In this way we get a contravariant functor $h_k : \mathbf{Mat} \rightarrow \mathbf{Set}$.

Now let $k, j \in \mathbb{N}^*$. A natural transformation $\alpha : h_k \rightarrow h_j$ consists of a family

$$(\alpha_n : h_k(n) = M_{n,k}(K) \rightarrow h_j(n) = M_{n,j}(K))_{n \in \mathbb{N}^*}$$

of functions such that for every arrow $A \in M_{m,n}(K)$ the following diagram is commutative:

$$\begin{array}{ccc} M_{n,k}(K) = h_k(n) & \xrightarrow{\alpha_n} & h_j(n) = M_{n,j}(K) \\ h_k(A) \downarrow & & \downarrow h_j(A) \\ M_{m,k}(K) = h_k(m) & \xrightarrow{\alpha_m} & h_j(m) = M_{m,j}(K) \end{array}$$

Let us classify all natural transformations $\alpha : h_k \rightarrow h_j$ or, in other words, all naturally defined column operations that transform k -column matrices to j -column matrices.

Now let us see how this setting matches the one of the Yoneda lemma.

The covariant Yoneda functor is the functor

$$y : \mathbf{Mat} \rightarrow \text{Fun}(\mathbf{Mat}^{\text{op}}, \mathbf{Set})$$

defined on objects n of \mathbf{Mat} by (the contravariant functor)

$$y(n) = \text{Hom}_{\mathbf{Mat}}(-, n) : \mathbf{Mat} \rightarrow \mathbf{Set}$$

and on arrows $A : m \rightarrow n$ in \mathbf{Mat} , that is, $A \in M_{m,n}(K)$ by (the natural transformation)

$$y(A) = \text{Hom}_{\mathbf{Mat}}(-, A) : y(m) = \text{Hom}_{\mathbf{Mat}}(-, m) \rightarrow y(n) = \text{Hom}_{\mathbf{Mat}}(-, n).$$

Explicitly, we have the following.

- Let k be an object of \mathbf{Mat} . The contravariant functor $y(k) = \text{Hom}_{\mathbf{Mat}}(-, k) : \mathbf{Mat} \rightarrow \mathbf{Set}$ is defined on objects n of \mathbf{Mat} by

$$(y(k))(n) = \text{Hom}_{\mathbf{Mat}}(n, k) = M_{n,k}(K)$$

and on arrows $A \in \text{Hom}_{\mathbf{Mat}}(m, n) = M_{m,n}(K)$ in \mathbf{Mat} by

$$(y(k))(A) = \text{Hom}_{\mathbf{Mat}}(A, k) : \text{Hom}_{\mathbf{Mat}}(n, k) = M_{n,k}(K) \rightarrow \text{Hom}_{\mathbf{Mat}}(m, k) = M_{m,k}(K),$$

where for every $C \in \text{Hom}_{\mathbf{Mat}}(n, k) = M_{n,k}(K)$ we have:

$$[(y(k))(A)](C) = [\text{Hom}_{\mathbf{Mat}}(A, k)](C) = A \cdot C.$$

Hence we have the functor $y(k) = h_k : \mathbf{Mat} \rightarrow \mathbf{Set}$.

- Let $B \in M_{k,j}(K)$ be an arrow in \mathbf{Mat} . The component of the natural transformation

$$y(B) = \text{Hom}_{\mathbf{Mat}}(-, B) : \text{Hom}_{\mathbf{Mat}}(-, k) = h_k \rightarrow \text{Hom}_{\mathbf{Mat}}(-, j) = h_j$$

at an object n of \mathbf{Mat} is defined by

$$y(B)_n = \text{Hom}_{\mathbf{Mat}}(n, B) : \text{Hom}_{\mathbf{Mat}}(n, k) = M_{n,k}(K) \rightarrow \text{Hom}_{\mathbf{Mat}}(n, j) = M_{n,j}(K),$$

where for every $C \in \text{Hom}_{\mathbf{Mat}}(n, k) = M_{n,k}(K)$ we have:

$$[y(B)_n](C) = [\text{Hom}_{\mathbf{Mat}}(n, B)](C) = C \cdot B.$$

Hence we have the natural transformation $y(B) = \alpha : h_k \rightarrow h_j$.

By the Yoneda lemma we have:

$$\text{Hom}(h_k, h_j) = \text{Hom}(\text{Hom}_{\mathbf{Mat}}(-, k), \text{Hom}_{\mathbf{Mat}}(-, j)) \cong \text{Hom}_{\mathbf{Mat}}(k, j) = M_{k,j}(K).$$

Also using the proof of the Yoneda lemma, we obtain the following:

- Every naturally-defined column operation $\alpha : h_k \rightarrow h_j$ is determined by a single matrix $B \in M_{k,j}(K)$.
- We have $B = \alpha_k(I_k) \in h_j(k) = M_{k,j}(K)$, that is, B is obtained by applying the column operation

$$\alpha_k : h_k(k) = M_{k,k}(K) \rightarrow h_j(k) = M_{k,j}(K)$$

to the identity matrix I_k .

- The column operation

$$\alpha_n : h_k(n) = M_{n,k}(K) \rightarrow h_j(n) = M_{n,j}(K)$$

is given by right multiplication by the matrix $B = \alpha_k(I_k)$, that is, $\alpha_n(C) = C \cdot B$ for every matrix $C \in M_{n,k}(K)$.

- Every matrix $B \in M_{k,j}(K)$ determines a naturally-defined column operation defined by right multiplication.

Other consequences are the following:

- A naturally-defined column operation $\alpha : h_k \rightarrow h_k$ is invertible if and only if the matrix $B = \alpha_k(I_k) \in M_k(K)$ is invertible.

Recall the **elementary operations** on matrices from linear algebra: interchange two columns, multiply a column by a non-zero scalar, multiply a column by a scalar and add it to another column. They are invertible, because the corresponding **elementary matrices** are invertible.

- The composite $\beta \circ \alpha : h_k \rightarrow h_m$ of two naturally-defined column operations $\beta : h_j \rightarrow h_m$ and $\alpha : h_k \rightarrow h_j$ is defined by the right multiplication of the product of the corresponding matrices:

$$(\beta \circ \alpha)_k(I_k) = \alpha_k(I_k) \cdot \beta_j(I_j).$$

In this way elementary column operations generate all invertible column operations.

6.3 Limits and Colimits in Categories of Diagrams

Definition 6.4. A category \mathcal{E} is called **complete** if it has all small limits, that is, for any small category \mathcal{J} and functor $F : \mathcal{J} \rightarrow \mathcal{E}$, there is a limit for F .

Dually, a category \mathcal{E} is called **cocomplete** if it has all small colimits, that is, for any small category \mathcal{J} and functor $F : \mathcal{J} \rightarrow \mathcal{E}$, there is a colimit for F .

Proposition 6.8. For any locally small category \mathcal{C} , the functor category $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ is complete. Moreover, for a small category \mathcal{J} and a functor $F : \mathcal{J} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$, the limit of F is computed pointwise in the sense that, for every object C of \mathcal{C} , we have:

$$(\lim_{\underset{j \in \mathcal{J}}{\downarrow}} F_j)(C) = \lim_{\underset{j \in \mathcal{J}}{\downarrow}} (F_j(C)).$$

Proof. Let \mathcal{J} be a small category and let $F : \mathcal{J} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ be a functor. Assume that a limit for F does exist. Then it is an object of $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$, hence it is a functor $L = \lim_{\underset{j \in \mathcal{J}}{\downarrow}} F_j : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$.

By the Yoneda lemma and the fact that representable functors preserve limits, for every object C of \mathcal{C} we have the following (natural) isomorphisms in \mathbf{Set} :

$$L(C) \cong \text{Hom}(y(C), L) \cong \text{Hom}(y(C), \lim_{\underset{j \in \mathcal{J}}{\downarrow}} F_j) \cong \lim_{\underset{j \in \mathcal{J}}{\downarrow}} \text{Hom}(y(C), F_j) \cong \lim_{\underset{j \in \mathcal{J}}{\downarrow}} F_j(C).$$

Hence the limit should be defined by $(\lim_{\underset{j \in \mathcal{J}}{\downarrow}} F_j)(C) = \lim_{\underset{j \in \mathcal{J}}{\downarrow}} F_j(C)$. One may show that this definition yields a limit for F . □

Dually, we have the following result.

Proposition 6.9. For any locally small category \mathcal{C} , the functor category $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ is cocomplete. Moreover, for a small category \mathcal{J} and a functor $F : \mathcal{J} \rightarrow \text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$, the colimit of F is computed pointwise in the sense that, for every object C of \mathcal{C} , we have:

$$(\lim_{\underset{j \in \mathcal{J}}{\uparrow}} F_j)(C) = \lim_{\underset{j \in \mathcal{J}}{\uparrow}} (F_j(C)).$$

Remark 6.10. In particular, for any locally small category \mathcal{C} , the functor category $\text{Fun}(\mathcal{C}^{\text{op}}, \mathbf{Set})$ has products and coproducts, equalizers and coequalizers, terminal and initial objects, pullbacks and pushouts. Hence the functor category may have much richer properties than the original category \mathcal{C} . In order to prove a property in \mathcal{C} one may use the Yoneda embedding to go to the associated functor category, prove the property in the functor category, and finally use the Yoneda lemma to get the corresponding property in \mathcal{C} .

Chapter 7

Adjoint

“Adjoint functors arise everywhere.”

—Mac Lane

We assume that our categories are locally small.

7.1 Preliminary Definition

Definition 7.1. An **adjunction** or **adjoint pair** between categories \mathcal{C} and \mathcal{D} consists of two functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $U : \mathcal{D} \rightarrow \mathcal{C}$ and a natural transformation $\eta : 1_{\mathcal{C}} \rightarrow U \circ F$ with the following universal mapping property: for every objects C of \mathcal{C} and D of \mathcal{D} and for every arrow $f : C \rightarrow U(D)$, there is a unique arrow $g : F(C) \rightarrow D$ in \mathcal{D} such that the following diagram is commutative:

$$\begin{array}{ccc} U(F(C)) & \xrightarrow{U(g)} & U(D) \\ \eta_C \uparrow & \nearrow f & \\ C & & \end{array}$$

hence $U(g) \circ \eta_C = f$.

Here F is called a **left adjoint** of U , U is called a **right adjoint** of F and η is called the **unit of adjunction**. Also, (F, U) is called an **adjunction**. Sometimes, one denotes $F \dashv U$.

Remark 7.1. For every objects C of \mathcal{C} and D of \mathcal{D} , we may consider

$$\Phi : \text{Hom}_{\mathcal{D}}(F(C), D) \rightarrow \text{Hom}_{\mathcal{C}}(C, U(D))$$

given by $\Phi(g) = U(g) \circ \eta_C$ for every $g \in \text{Hom}_{\mathcal{D}}(F(C), D)$. By universal mapping property of adjunction, Φ is bijective.

Example 7.1. Let \mathcal{C} be a category and let $\Delta : \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ be the **diagonal functor** defined on objects C of \mathcal{C} by $\Delta(C) = (C, C)$ and on arrows $f : C \rightarrow C'$ in \mathcal{C} by $\Delta(f) = (f, f) : \Delta(C) = (C, C) \rightarrow \Delta(C') = (C', C')$. We are looking for a right adjoint of Δ , say $R : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$. Then we must have an isomorphism in **Set**:

$$\text{Hom}_{\mathcal{C}}(C, R(X, Y)) \cong \text{Hom}_{\mathcal{C} \times \mathcal{C}}(\Delta(C), (X, Y))$$

for every object C of \mathcal{C} and for every object $(X, Y) \in \mathcal{C} \times \mathcal{C}$.

We define the functor $R : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$ on objects (X, Y) of $\mathcal{C} \times \mathcal{C}$ by $R(X, Y) = X \times Y$ and on arrows (f, g) in $\mathcal{C} \times \mathcal{C}$ by $R(f, g) = f \times g$. One shows that (Δ, R) is an adjunction with unit $\eta : 1_{\mathcal{C}} \rightarrow R \circ \Delta$, whose component at an object C of \mathcal{C} is

$$\eta_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix} : C \rightarrow (R \circ \Delta)(C) = C \times C.$$

For every arrow $h : C \rightarrow D$ in \mathcal{C} , the following diagram is commutative:

$$\begin{array}{ccc} C & \xrightarrow{\eta_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}} & (R \circ \Delta)(C) = C \times C \\ h \downarrow & & \downarrow h \times h = \begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} \\ D & \xrightarrow{\eta_D = \begin{bmatrix} 1 \\ 1 \end{bmatrix}} & (R \circ \Delta)(D) = D \times D \end{array}$$

hence η is a natural transformation.

Next one checks universal mapping property for η , namely for every objects C of \mathcal{C} and (X, Y) of $\mathcal{C} \times \mathcal{C}$ and for every arrow $f : C \rightarrow R(X, Y) = X \times Y$ in \mathcal{C} , there is a unique arrow $g : \Delta(C) = (C, C) \rightarrow (X, Y)$ in $\mathcal{C} \times \mathcal{C}$ such that the following diagram is commutative:

$$\begin{array}{ccc} R(\Delta(C)) = C \times C & \xrightarrow{R(g)} & R(X, Y) = X \times Y \\ \eta_C \uparrow & \nearrow f & \\ C & & \end{array}$$

hence $R(g) \circ \eta_C = f$. Since $f : C \rightarrow X \times Y$, it is of the form $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, where $f_1 : C \rightarrow X$ and $f_2 : C \rightarrow Y$. Then we may choose

$$g = (f_1, f_2) : (C, C) \rightarrow (X, Y)$$

One checks that this choice yields the required properties.

7.2 Hom-set Definition

Theorem 7.2. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $U : \mathcal{D} \rightarrow \mathcal{C}$ be functors. Then (F, U) is called an adjunction if one of the following equivalent conditions holds:

- (1) (preliminary definition) There exists a natural transformation $\eta : 1_{\mathcal{C}} \rightarrow U \circ F$ (called the **unit of adjunction**) with the following universal mapping property: for every objects C of \mathcal{C} and D of \mathcal{D} and for every arrow $f : C \rightarrow U(D)$, there is a unique arrow $g : F(C) \rightarrow D$ in \mathcal{D} such that the following diagram is commutative:

$$\begin{array}{ccc} U(F(C)) & \xrightarrow{U(g)} & U(D) \\ \eta_C \uparrow & \nearrow f & \\ C & & \end{array}$$

hence $U(g) \circ \eta_C = f$.

- (2) (official definition) For every objects C of \mathcal{C} and D of \mathcal{D} there is an isomorphism in **Set**:

$$\Phi_{C,D} : \text{Hom}_{\mathcal{D}}(F(C), D) \rightarrow \text{Hom}_{\mathcal{C}}(C, U(D))$$

which is natural in both C and D .

- (3) (dual to preliminary definition) There exists a natural transformation $\varepsilon : F \circ U \rightarrow 1_{\mathcal{D}}$ (called the **counit of adjunction**) with the following universal mapping property: for every objects C of \mathcal{C} and D of \mathcal{D} and for every arrow $g : F(C) \rightarrow D$, there is a unique arrow $f : C \rightarrow U(D)$ in \mathcal{C} such that the following diagram is commutative:

$$\begin{array}{ccc} F(U(D)) & \xleftarrow{F(f)} & F(C) \\ \varepsilon_D \downarrow & \nwarrow g & \\ D & & \end{array}$$

hence $\varepsilon_D \circ F(f) = g$.

If the inverse of $\Phi_{C,D} : \text{Hom}_{\mathcal{D}}(F(C), D) \rightarrow \text{Hom}_{\mathcal{C}}(C, U(D))$ is

$$\Psi_{C,D} : \text{Hom}_{\mathcal{C}}(C, U(D)) \rightarrow \text{Hom}_{\mathcal{D}}(F(C), D),$$

we have the following relating formulas:

- $\Phi_{C,D}(g) = U(g) \circ \eta_C$ and $\eta_C = \Phi_{C,F(C)}(1_{F(C)})$.
- $\Psi_{C,D}(f) = \varepsilon_D \circ F(f)$ and $\varepsilon_D = \Psi_{U(D),D}(1_{U(D)})$.

Proof. (1) \implies (2) Define $\Phi_{C,D}(g) = U(g) \circ \eta_C$. By universal mapping property from (1) we know that $\Phi_{C,D}$ is an isomorphism in **Set** (bijective).

We show naturality in C . Let $h : C' \rightarrow C$ be an arrow in \mathcal{C} . We need to prove that the following diagram is commutative:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(F(C), D) & \xrightarrow{\Phi_{C,D}} & \text{Hom}_{\mathcal{C}}(C, U(D)) \\ \text{Hom}_{\mathcal{D}}(F(h), D) \downarrow & & \downarrow \text{Hom}_{\mathcal{C}}(h, U(D)) \\ \text{Hom}_{\mathcal{D}}(F(C'), D) & \xrightarrow{\Phi_{C',D}} & \text{Hom}_{\mathcal{C}}(C', U(D)) \end{array}$$

Denote $h^* = \text{Hom}_{\mathcal{C}}(h, U(D))$ and $F(h)^* = \text{Hom}_{\mathcal{D}}(F(h), D)$. Let $g \in \text{Hom}_{\mathcal{D}}(F(C), D)$. We have:

$$(h^* \circ \Phi_{C,D})(g) = h^*(U(g) \circ \eta_C) = U(g) \circ \eta_C \circ h.$$

Since η is a natural transformation, the following diagram is commutative:

$$\begin{array}{ccc} C' & \xrightarrow{\eta_{C'}} & (U \circ F)(C') \\ h \downarrow & & \downarrow (U \circ F)(h) \\ C & \xrightarrow{\eta_C} & (U \circ F)(C) \end{array}$$

It follows that:

$$\begin{aligned} (h^* \circ \Phi_{C,D})(g) &= U(g) \circ (U \circ F)(h) \circ \eta_{C'} = U(g \circ F(h)) \circ \eta_{C'} \\ &= \Phi_{C',D}(g \circ F(h)) = \Phi_{C',D}(F(h)^*(g)) = (\Phi_{C',D} \circ F(h)^*)(g). \end{aligned}$$

Hence we have naturality in C .

We show naturality in D . Let $k : D \rightarrow D'$ be an arrow in \mathcal{D} . We need to prove that the following diagram is commutative:

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(F(C), D) & \xrightarrow{\Phi_{C,D}} & \text{Hom}_{\mathcal{C}}(C, U(D)) \\ \text{Hom}_{\mathcal{D}}(F(C), k) \downarrow & & \downarrow \text{Hom}_{\mathcal{C}}(C, U(k)) \\ \text{Hom}_{\mathcal{D}}(F(C), D') & \xrightarrow{\Phi_{C,D'}} & \text{Hom}_{\mathcal{C}}(C, U(D')) \end{array}$$

Denote $k_* = \text{Hom}_{\mathcal{D}}(F(C), k)$ and $U(k)_* = \text{Hom}_{\mathcal{C}}(C, U(k))$. Let $g \in \text{Hom}_{\mathcal{D}}(F(C), D)$. We have:

$$\begin{aligned} (U(k)_* \circ \Phi_{C,D})(g) &= U(k)_*(U(g) \circ \eta_C) = U(k) \circ U(g) \circ \eta_C = U(k \circ g) \circ \eta_C \\ &= \Phi_{C,D'}(k \circ g) = \Phi_{C,D'}(k_*(g)) = (\Phi_{C,D'} \circ k_*)(g). \end{aligned}$$

Hence we have naturality in D .

(2) \implies (1) We have seen that naturality in D means that for every arrow $k : D \rightarrow D'$ in \mathcal{D} we have:

$$(\Phi_{C,D'} \circ k_*)(g) = (U(k)_* \circ \Phi_{C,D})(g), \quad \forall g \in \text{Hom}_{\mathcal{D}}(F(C), D),$$

that is,

$$\Phi_{C,D'}(k \circ g) = U(k) \circ \Phi_{C,D}(g), \quad \forall g \in \text{Hom}_{\mathcal{D}}(F(C), D). \quad (7.1)$$

Also, naturality in C means that for every arrow $h : C' \rightarrow C$ in \mathcal{C} we have:

$$(h^* \circ \Phi_{C,D})(g) = (\Phi_{C',D} \circ F(h)^*)(g), \quad \forall g \in \text{Hom}_{\mathcal{D}}(F(C), D),$$

that is,

$$\Phi_{C,D}(g) \circ h = \Phi_{C',D}(g \circ F(h)), \quad \forall g \in \text{Hom}_{\mathcal{D}}(F(C), D). \quad (7.2)$$

We show that $\eta : 1_{\mathcal{C}} \rightarrow U \circ F$ with the following components at an object C of \mathcal{C} :

$$\eta_C : C \rightarrow (U \circ F)(C), \quad \eta_C = \Phi_{C,F(C)}(1_{F(C)})$$

is a natural transformation. Let $h : C' \rightarrow C$ be an arrow in \mathcal{C} . Consider the following diagram:

$$\begin{array}{ccc} C' & \xrightarrow{\eta_{C'}} & (U \circ F)(C') \\ h \downarrow & & \downarrow (U \circ F)(h) \\ C & \xrightarrow{\eta_C} & (U \circ F)(C) \end{array}$$

We have:

$$\begin{aligned} (U \circ F)(h) \circ \eta_{C'} &= (U \circ F)(h) \circ \Phi_{C',F(C')} (1_{F(C')}) = U(F(h)) \circ \Phi_{C',F(C')} (1_{F(C')}) \\ &\stackrel{(7.1)}{=} \Phi_{C',F(C)} (F(h) \circ 1_{F(C')}) = \Phi_{C',F(C)} (1_{F(C)} \circ F(h)) \\ &\stackrel{(7.2)}{=} \Phi_{C,F(C)} (1_{F(C)}) \circ h = \eta_C \circ h, \end{aligned}$$

where we have used (7.1) for $g = 1_{F(C')}$, $k = F(h) : F(C') \rightarrow F(C)$ and (7.2) for $g = 1_{F(C)}$, $h : C' \rightarrow C$. Hence the above diagram is commutative, and thus η is a natural transformation.

Let $f \in \text{Hom}_{\mathcal{C}}(C, U(D))$. Then there is a unique arrow $g \in \text{Hom}_{\mathcal{D}}(F(C), D)$ such that $f = \Phi_{C,D}(g)$. But we have:

$$U(g) \circ \eta_C = U(g) \circ \Phi_{C,F(C)} (1_{F(C)}) = \Phi_{C,D} (g \circ 1_{F(C)}) = \Phi_{C,D}(g),$$

hence $f = \Phi_{C,D}(g) = U(g) \circ \eta_C$. This shows the required universal mapping property for η .

(2) \iff (3) This follows by the duality principle, because we have (1) \iff (2). \square

7.3 Examples of Adjoints

Proposition 7.3. Adjoints are unique up to an isomorphism, that is, given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ and functors $U, V : \mathcal{D} \rightarrow \mathcal{C}$ such that (F, U) and (F, V) are adjoint pairs, then $U \cong V$.

Proof. For every objects C of \mathcal{C} and D of \mathcal{D} , we have the following isomorphisms in **Set**, natural in both C and D :

$$\text{Hom}_{\mathcal{C}}(C, U(D)) \cong \text{Hom}_{\mathcal{D}}(F(C), D) \cong \text{Hom}_{\mathcal{C}}(C, V(D))$$

Then by the Yoneda lemma, we deduce that $U(D) \cong V(D)$. By adjointness, this isomorphism is again natural in D . Hence we have $U \cong V$. \square

Example 7.2. Consider the following functors:

- The **free** functor $F : \mathbf{Set} \rightarrow \mathbf{Mon}$ defined by $F(A) = A^*$ for every set A , where A^* is the **free monoid on the set** A (i.e. A^* is the set of “words” with “letters” from A , and the operation on A^* is the concatenation of “words”), and for every function $f : A \rightarrow B$, $F(f) = f^* = \bar{f} : F(A) = A^* \rightarrow F(B) = B^*$, where \bar{f} is the unique monoid homomorphism with $\bar{f} \circ i_A = i_B \circ f$ and $i_A : A \rightarrow A^*$, $i_B : B \rightarrow B^*$ are the inclusion monoid homomorphisms.
- The forgetful functor $U : \mathbf{Mon} \rightarrow \mathbf{Set}$.

Let us show that (F, U) is an adjunction.

By universal mapping property of the free monoid, for every monoid (B, \cdot) and for every function $f : A \rightarrow B$, there is a unique monoid homomorphism $\bar{f} : A^* \rightarrow B$ such that $\bar{f} \circ i_A = f$.

Let A be a set and let (B, \cdot) be a monoid. We prove that there is an isomorphism in **Set** (bijection):

$$\Phi_{A,B} : \text{Hom}_{\mathbf{Mon}}(F(A), B) \rightarrow \text{Hom}_{\mathbf{Set}}(A, U(B)),$$

that is,

$$\Phi_{A,B} : \text{Hom}_{\mathbf{Mon}}(A^*, B) \rightarrow \text{Hom}_{\mathbf{Set}}(A, B).$$

For every $g \in \text{Hom}_{\mathbf{Mon}}(A^*, B)$, we define $\Phi_{A,B}(g) = g \circ i_A$. Let us also consider

$$\Psi_{A,B} : \text{Hom}_{\mathbf{Set}}(A, B) \rightarrow \text{Hom}_{\mathbf{Mon}}(A^*, B)$$

defined for every $f \in \text{Hom}_{\mathbf{Set}}(A, B)$ by $\Psi_{A,B}(f) = \bar{f}$, where $\bar{f} \circ i_A = f$.

Note that $\bar{f} : A^* \rightarrow B$ is defined by $\bar{f}(a_1 \dots a_n) = f(a_1) \cdots f(a_n)$.

For every $f \in \text{Hom}_{\mathbf{Set}}(A, B)$ we have:

$$\Phi_{A,B}(\Psi_{A,B}(f)) = \Phi_{A,B}(\bar{f}) = \bar{f} \circ i_A = f.$$

For every $g \in \text{Hom}_{\mathbf{Mon}}(A^*, B)$ we have:

$$\Psi_{A,B}(\Phi_{A,B}(g)) = \Psi_{A,B}(g \circ i_A) = \overline{g \circ i_A}.$$

For every $w = a_1 \dots a_n \in A^*$, we have:

$$\overline{g \circ i_A}(w) = (g \circ i_A)(a_1) \cdots (g \circ i_A)(a_n) = g(a_1) \cdots g(a_n) = g(w).$$

Hence $\Psi_{A,B}(\Phi_{A,B}(g)) = g$. Therefore, $\Phi_{A,B}$ is a bijection.

We prove that $\Phi_{A,B}$ is natural in A . Let $\alpha : A \rightarrow A'$ be a function. Then $\alpha^* : A^* \rightarrow A'^*$. Consider the following diagram:

$$\begin{array}{ccc} \text{Hom}_{\mathbf{Mon}}(A'^*, B) & \xrightarrow{\Phi_{A',B}} & \text{Hom}_{\mathbf{Set}}(A', B) \\ \text{Hom}_{\mathbf{Mon}}(\alpha^*, B) \downarrow & & \downarrow \text{Hom}_{\mathbf{Set}}(\alpha, B) \\ \text{Hom}_{\mathbf{Mon}}(A^*, B) & \xrightarrow{\Phi_{A,B}} & \text{Hom}_{\mathbf{Set}}(A, B) \end{array}$$

For every $g \in \text{Hom}_{\mathbf{Mon}}(A^*, B)$, we have:

$$\begin{aligned} [\Phi_{A,B} \circ \text{Hom}_{\mathbf{Mon}}(\alpha^*, B)](g) &= \Phi_{A,B}(g \circ \alpha^*) = g \circ \alpha^* \circ i_A = g \circ \bar{\alpha} \circ i_A \\ &= g \circ i_{A'} \circ \alpha = \text{Hom}_{\mathbf{Set}}(\alpha, B)(g \circ i_{A'}) \\ &= [\text{Hom}_{\mathbf{Set}}(\alpha, B) \circ \Phi_{A',B}](g). \end{aligned}$$

Hence the above diagram is commutative, and thus $\Phi_{A,B}$ is natural in A .

We prove that $\Phi_{A,B}$ is natural in B . Let $\beta : B \rightarrow B'$ be a monoid homomorphism. Consider the following diagram:

$$\begin{array}{ccc} \text{Hom}_{\mathbf{Mon}}(A^*, B) & \xrightarrow{\Phi_{A,B}} & \text{Hom}_{\mathbf{Set}}(A, B) \\ \text{Hom}_{\mathbf{Mon}}(A^*, \beta) \downarrow & & \downarrow \text{Hom}_{\mathbf{Set}}(A, \beta) \\ \text{Hom}_{\mathbf{Mon}}(A^*, B') & \xrightarrow{\Phi_{A,B'}} & \text{Hom}_{\mathbf{Set}}(A, B') \end{array}$$

For every $g \in \text{Hom}_{\mathbf{Mon}}(A^*, B)$, we have:

$$\begin{aligned} [\Phi_{A,B'} \circ \text{Hom}_{\mathbf{Mon}}(A^*, \beta)](g) &= \Phi_{A,B'}(\beta \circ g) = \beta \circ g \circ i_A = \beta \circ \Phi_{A,B}(g) \\ &= [\text{Hom}_{\mathbf{Set}}(A, \beta) \circ \Phi_{A,B}](g). \end{aligned}$$

Hence the above diagram is commutative, and thus $\Phi_{A,B}$ is natural in B .

Therefore, (F, U) is an adjunction.

Alternatively, one may show that there is the natural transformation $\eta : \mathbf{1}_{\mathbf{Set}} \rightarrow U \circ F$, whose component at a set A is

$$\eta_A = i_A : A \rightarrow (U \circ F)(A) = A^*,$$

and prove that it satisfies the required universal mapping property of adjunction. Note that

$$\eta_A = \Phi_{A,F(A)}(1_{F(A)}) = \Phi_{A,F(A)}(1_{A^*}) = i_A : A \rightarrow A^*.$$

Example 7.3. Consider the category \mathcal{T} of all torsion abelian groups (i.e., abelian groups in which every element has finite order) and group homomorphisms. Consider the following functors:

- The inclusion functor $I : \mathcal{T} \rightarrow \mathbf{Ab}$.
- The **torsion** functor $T : \mathbf{Ab} \rightarrow \mathcal{T}$ defined by $T(G) = t(G) = \{x \in G \mid x \text{ has finite order}\}$ for every abelian group G , and $T(f) = f|_{t(G)} : t(G) \rightarrow t(H)$ for every group homomorphism $f : G \rightarrow H$.

Let us show that (I, T) is an adjunction.

Note that $T(f) = f|_{t(G)} : t(G) \rightarrow t(H)$ is well defined, because if the order of an element $x \in G$ is finite, then the order of $f(x)$ is also finite.

We look for a natural transformation $\eta : 1_{\mathcal{T}} \rightarrow T \circ I$. For every object G of \mathcal{T} , we define

$$\eta_G = 1_G : G \rightarrow (T \circ I)(G) = t(G) = G.$$

Clearly, η is a natural transformation. We show universal mapping property for η . Let G be an object of \mathcal{T} , H an object of \mathbf{Ab} and $f : G \rightarrow T(H)$ a group homomorphism. We look for a unique group homomorphism $g : I(G) = G \rightarrow H$ such that $T(g) \circ \eta_G = f$. We have:

$$T(g) \circ \eta_G = T(g) = g|_{t(G)} = g|_G = g.$$

In fact, we have $g : G \rightarrow t(H)$, and we may take $g = f$.

Therefore, (I, T) is an adjunction.

Alternatively, for every torsion abelian group A and for every abelian group B , one proves that there is an isomorphism in **Set** (bijection) $\Phi_{A,B} : \text{Hom}_{\mathbf{Ab}}(I(A), B) \rightarrow \text{Hom}_{\mathcal{T}}(A, T(B))$, that is, $\Phi_{A,B} : \text{Hom}_{\mathbf{Ab}}(A, B) \rightarrow \text{Hom}_{\mathcal{T}}(A, t(B))$. For every $g \in \text{Hom}_{\mathbf{Ab}}(I(A), B) = \text{Hom}_{\mathbf{Ab}}(A, B)$, we have

$$\Phi_{A,B}(g) = T(g) \circ \eta_A = T(g) \circ 1_A = T(g) = g|_{t(A)} = g|_A = g.$$

In fact, we have $g : A \rightarrow t(B)$, hence $\Phi_{A,B}(g) = g \in \text{Hom}_{\mathcal{T}}(A, t(B))$.

Example 7.4. Let (A, \leq) and (B, \leq) be posets, and let $f : A \rightarrow B$ and $g : B \rightarrow A$ be order-preserving functions (i.e. functors between poset categories). The following conditions are equivalent:

- (i) For every $a \in A$ and $b \in B$, $f(a) \leq b \Leftrightarrow a \leq g(b)$.
- (ii) For every $a \in A$, $a \leq g(f(a))$, and for every $b \in B$, $f(g(b)) \leq b$.

In this case (f, g) is called a **monotone Galois connection**. In fact, (f, g) is an adjunction between the poset categories (A, \leq) and (B, \leq) .

Example 7.5. Let Ω^X be the set of propositional functions on a set X , that is, functions $P : X \rightarrow \Omega = \{\perp, \top\}$, interpreted as declaring, for each $x \in X$, whether $P(x)$ is false or true. Consider the poset categories (Ω, \leq) , where $\perp \leq \top$, and (Ω^X, \leq) , where $P \leq Q$ if and only if $P(x) \leq Q(x)$ for all $x \in X$, which is the case if and only if P implies Q . Consider the following functors:

- $\exists_X : \Omega^X \rightarrow \Omega$ defined by $\exists_X(P) = \top$ if and only if there is $x \in X$ with $P(x) = \top$.
- The constant “dummy variable” functor $\Delta_X : \Omega \rightarrow \Omega^X$.
- $\forall_X : \Omega^X \rightarrow \Omega$ defined by $\forall_X(P) = \top$ if and only if $P(x) = \top$ for all $x \in X$.

Then (\exists_X, Δ_X) and (Δ_X, \forall_X) are adjunctions.

7.4 Further Properties of Adjoints

Proposition 7.4. Let $L_1 : \mathcal{A} \rightarrow \mathcal{B}$, $L_2 : \mathcal{B} \rightarrow \mathcal{C}$, $R_1 : \mathcal{B} \rightarrow \mathcal{A}$ and $R_2 : \mathcal{C} \rightarrow \mathcal{B}$ be functors such that (L_1, R_1) and (L_2, R_2) are adjoint pairs. Then $(L_2 \circ L_1, R_1 \circ R_2)$ is an adjoint pair.

Proof. For every objects A of \mathcal{A} and C of \mathcal{C} we have the natural isomorphism in **Set**:

$$\begin{aligned} \text{Hom}_{\mathcal{C}}((L_2 \circ L_1)(A), C) &\cong \text{Hom}_{\mathcal{C}}(L_2(L_1(A)), C) \cong \text{Hom}_{\mathcal{B}}(L_1(A), R_2(C)) \\ &\cong \text{Hom}_{\mathcal{A}}(A, R_1(R_2(C))) \cong \text{Hom}_{\mathcal{A}}(A, (R_1 \circ R_2)(C)). \end{aligned}$$

Hence $(L_2 \circ L_1, R_1 \circ R_2)$ is an adjoint pair. \square

Proposition 7.5. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $U : \mathcal{D} \rightarrow \mathcal{C}$ be functors. The following are equivalent:

- (1) (F, U) is an adjunction.
- (2) There are natural transformations $\eta : 1_{\mathcal{C}} \rightarrow U \circ F$ and $\varepsilon : F \circ U \rightarrow 1_{\mathcal{D}}$ such that for every objects C of \mathcal{C} and D of \mathcal{D} the following composite functors yield identities on $F(C)$ and $U(D)$ respectively:

$$\begin{aligned} F(C) &\xrightarrow{F(\eta_C)} (F \circ U \circ F)(C) \xrightarrow{\varepsilon_{F(C)}} F(C), \\ U(D) &\xrightarrow{\eta_{U(D)}} (U \circ F \circ U)(D) \xrightarrow{U(\varepsilon_D)} U(D), \end{aligned}$$

that is, $\varepsilon_{F(C)} \circ F(\eta_C) = 1_{F(C)}$ and $U(\varepsilon_D) \circ \eta_{U(D)} = 1_{U(D)}$. They are shortly written as $\varepsilon F \circ F \eta = 1_F$ and $U \varepsilon \circ \eta U = 1_U$, and are called the triangle identities.

Proof. (1) \implies (2) Assume that (F, U) is an adjunction. By Theorem 7.2, for every objects C of \mathcal{C} and D of \mathcal{D} there is an isomorphism $\Phi_{C,D} : \text{Hom}_{\mathcal{D}}(F(C), D) \rightarrow \text{Hom}_{\mathcal{C}}(C, U(D))$ in **Set** with inverse $\Psi_{C,D} : \text{Hom}_{\mathcal{C}}(C, U(D)) \rightarrow \text{Hom}_{\mathcal{D}}(F(C), D)$.

By Theorem 7.2, there exists a natural transformation $\varepsilon : F \circ U \rightarrow 1_{\mathcal{D}}$ with the following universal mapping property: for every objects C of \mathcal{C} and D of \mathcal{D} and for every arrow $g : F(C) \rightarrow D$, there is a unique arrow $f : C \rightarrow U(D)$ in \mathcal{C} such that $\varepsilon_D \circ F(f) = g$. In particular, for $D = F(C)$ and $g = 1_{F(C)}$, using a relating formula from Theorem 7.2, there is a unique arrow $f : C \rightarrow U(F(C))$ such that $\Psi_{C,F(C)}(f) = \varepsilon_{F(C)} \circ F(f) = 1_{F(C)}$. Then

$$f = \Phi_{C,F(C)}(\Psi_{C,F(C)}(f)) = \Phi_{C,F(C)}(1_{F(C)}) = \eta_C,$$

and thus $\varepsilon_{F(C)} \circ F(\eta_C) = 1_{F(C)}$.

By Theorem 7.2, there exists a natural transformation $\eta : 1_{\mathcal{C}} \rightarrow U \circ F$ with the following universal mapping property: for every objects C of \mathcal{C} and D of \mathcal{D} and for every arrow $f : C \rightarrow U(D)$, there is a unique arrow $g : F(C) \rightarrow D$ in \mathcal{D} such that $U(g) \circ \eta_C = f$. In particular, for $C = U(D)$ and $f = 1_{U(D)}$, using a relating formula from Theorem 7.2, there is a unique arrow $g : F(U(D)) \rightarrow D$ such that $\Phi_{U(D),D}(g) = U(g) \circ \eta_{U(D)} = 1_{U(D)}$. Then

$$g = \Psi_{U(D),D}(\Phi_{U(D),D}(g)) = \Psi_{U(D),D}(1_{U(D)}) = \varepsilon_D,$$

and thus $U(\varepsilon_D) \circ \eta_{U(D)} = 1_{U(D)}$.

(2) \implies (1) Suppose that there are natural transformations $\eta : 1_{\mathcal{C}} \rightarrow U \circ F$ and $\varepsilon : F \circ U \rightarrow 1_{\mathcal{D}}$ such that for every objects C of \mathcal{C} and D of \mathcal{D} , we have $\varepsilon_{F(C)} \circ F(\eta_C) = 1_{F(C)}$ and $U(\varepsilon_D) \circ \eta_{U(D)} = 1_{U(D)}$.

We show that η has universal mapping property from Theorem 7.2. Let C be an object of \mathcal{C} , let D be an object of \mathcal{D} and let $f : C \rightarrow U(D)$ be an arrow in \mathcal{C} . We look for a unique arrow $g : F(C) \rightarrow D$ such that $U(g) \circ \eta_C = f$. Choose $g = \varepsilon_D \circ F(f)$. By the naturality of η , we have the following commutative diagram:

$$\begin{array}{ccc} C & \xrightarrow{\eta_C} & (U \circ F)(C) \\ f \downarrow & & \downarrow (U \circ F)(f) \\ U(D) & \xrightarrow{\eta_{U(D)}} & (U \circ F)(U(D)) \end{array}$$

hence $(U \circ F)(f) \circ \eta_C = \eta_{U(D)} \circ f$. It follows that:

$$U(g) \circ \eta_C = U(\varepsilon_D) \circ (U \circ F)(f) \circ \eta_C = U(\varepsilon_D) \circ \eta_{U(D)} \circ f = 1_{U(D)} \circ f = f.$$

For uniqueness, suppose that there is also an arrow $g' : F(C) \rightarrow D$ such that $U(g') \circ \eta_C = f = U(g) \circ \eta_C$. Then $F(U(g')) \circ F(\eta_C) = F(U(g)) \circ F(\eta_C)$. By the naturality of ε , we have the following commutative diagram:

$$\begin{array}{ccc} (F \circ U)(F(C)) & \xrightarrow{\varepsilon_{F(C)}} & F(C) \\ (F \circ U)(g) \downarrow & & \downarrow g \\ (F \circ U)(D) & \xrightarrow{\varepsilon_D} & D \end{array}$$

hence $g \circ \varepsilon_{F(C)} = \varepsilon_D \circ (F \circ U)(g)$, and a similar equality for g' . It follows that:

$$\begin{aligned} g' &= g' \circ 1_{F(C)} = g' \circ \varepsilon_{F(C)} \circ F(\eta_C) = \varepsilon_D \circ (F \circ U)(g') \circ F(\eta_C) \\ &= \varepsilon_D \circ F(U(g') \circ \eta_C) = \varepsilon_D \circ F(U(g) \circ \eta_C) = g \circ \varepsilon_{F(C)} \circ F(\eta_C) = g \circ 1_{F(C)} = g. \end{aligned}$$

Therefore, (F, U) is an adjunction by Theorem 7.2. \square

Corollary 7.6. Let (L, R) be an adjunction with $L : \mathcal{A} \rightarrow \mathcal{B}$, $R : \mathcal{B} \rightarrow \mathcal{A}$, unit $\eta : 1_{\mathcal{A}} \rightarrow R \circ L$ and counit $\varepsilon : L \circ R \rightarrow 1_{\mathcal{B}}$. Consider the following full subcategory of \mathcal{A} :

$$\text{Fix}(R \circ L) = \{A \in \mathcal{A} \mid \eta_A : A \rightarrow R(L(A)) \text{ is an isomorphism}\},$$

and the following full subcategory of \mathcal{B} :

$$\text{Fix}(L \circ R) = \{B \in \mathcal{B} \mid \varepsilon_B : L(R(B)) \rightarrow B \text{ is an isomorphism}\}$$

Then (L, R) restricts to an equivalence between the categories $\text{Fix}(R \circ L)$ and $\text{Fix}(L \circ R)$.

Proof. Let A be an object of $\text{Fix}(R \circ L)$. Then $\eta_A : A \rightarrow R(L(A))$ is an isomorphism. It follows that $L(\eta_A)$ is an isomorphism. By the previous proposition we have $\varepsilon_{L(A)} \circ L(\eta_A) = 1_{L(A)}$. It follows that $\varepsilon_{L(A)} : L(R(L(A))) \rightarrow L(A)$ is an isomorphism. This shows that $L(A) \in \text{Fix}(L \circ R)$. Hence we may consider the functor $L' : \text{Fix}(R \circ L) \rightarrow \text{Fix}(L \circ R)$, which is the restriction of the functor L to $\text{Fix}(R \circ L)$.

Also, let B be an object of $\text{Fix}(L \circ R)$. Then $\varepsilon_B : L(R(B)) \rightarrow B$ is an isomorphism. It follows that $R(\varepsilon_B)$ is an isomorphism. By the previous proposition we have $R(\varepsilon_B) \circ \eta_{R(B)} = 1_{R(B)}$. It follows that $\eta_{R(B)} : R(B) \rightarrow R(L(R(B)))$ is an isomorphism. This shows that $R(B) \in \text{Fix}(R \circ L)$. Hence we may consider the functor $R' : \text{Fix}(L \circ R) \rightarrow \text{Fix}(R \circ L)$, which is the restriction of the functor R to $\text{Fix}(L \circ R)$.

Clearly, the restrictions $\eta' : 1_{\text{Fix}(R \circ L)} \rightarrow R' \circ L'$ of η and $\varepsilon' : L' \circ R' \rightarrow 1_{\text{Fix}(L \circ R)}$ of ε are natural isomorphisms. Hence L' and R' define an equivalence between the categories $\text{Fix}(R \circ L)$ and $\text{Fix}(L \circ R)$. \square

Example 7.6. An equivalence of categories with associated natural isomorphisms η and ε need not be an adjunction with unit η and counit ε . For instance, let (M, \cdot) and (N, \cdot) be monoids, and view them as monoid categories. An equivalence of categories between them is a pair of functors, that is, monoid homomorphisms $f : M \rightarrow N$ and $g : N \rightarrow M$ and two natural isomorphisms $\alpha : 1_M \rightarrow g \circ f$ and $\beta : f \circ g \rightarrow 1_N$, that is, invertible elements $\alpha \in M$ and $\beta \in N$. The triangle identities imply $f(\alpha) = \beta$ and $g(\beta) = \alpha$. Taking $M = N$ and $f = g = 1_M$, an equivalence of categories satisfying the triangle identities consists of two central invertible elements $\alpha, \beta \in M$ such that $\alpha = \beta$. Now let (M, \cdot) be a non-trivial group and $\alpha, \beta \in M$ be distinct. Then (α, β) is an equivalence which is not an adjunction.

Proposition 7.7. If (F, U) is an equivalence of categories with $F : \mathcal{C} \rightarrow \mathcal{D}$, $U : \mathcal{D} \rightarrow \mathcal{C}$ and natural isomorphisms $\eta : 1_{\mathcal{C}} \rightarrow U \circ F$ and $\varepsilon : F \circ U \rightarrow 1_{\mathcal{D}}$, then there is a unique natural isomorphism $\eta_0 : 1_{\mathcal{C}} \rightarrow U \circ F$ such that (F, U) is an adjoint pair with unit η_0 and counit ε , and a unique natural isomorphism $\varepsilon_0 : F \circ U \rightarrow 1_{\mathcal{D}}$ such that (F, U) is an adjoint pair with unit η and counit ε_0 .

We say that every equivalence of categories can be turned into an **adjoint equivalence**.

Proposition 7.8 (RAPL, LAPC). Right adjoints preserve limits and left adjoints preserve colimits. In particular, right adjoints preserve products, terminal objects, equalizers and pullbacks, while left adjoints preserve coproducts, initial objects, coequalizers and pushouts.

Proof. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ and $U : \mathcal{D} \rightarrow \mathcal{C}$ be functors such that (F, U) is an adjoint pair. Let $D : \mathcal{J} \rightarrow \mathcal{D}$ be a diagram such that there is $\varprojlim D_j$ in \mathcal{D} . For every object X of \mathcal{C} we have the natural isomorphisms in **Set**:

$$\begin{aligned} \text{Hom}_{\mathcal{C}}(X, U(\varprojlim D_j)) &\cong \text{Hom}_{\mathcal{D}}(F(X), \varprojlim D_j) \cong \varprojlim \text{Hom}_{\mathcal{D}}(F(X), D_j) \\ &\cong \varprojlim \text{Hom}_{\mathcal{C}}(X, U(D_j)) \cong \text{Hom}_{\mathcal{C}}(X, \varprojlim U(D_j)), \end{aligned}$$

where we have used the adjunction isomorphisms and the fact that the functor $\text{Hom}_{\mathcal{D}}(F(C), -)$ preserves limits. The by the Yoneda lemma, we deduce that $U(\varprojlim D_j) \cong \varprojlim U(D_j)$. \square

Example 7.7. (1) The above property may be useful in practice to show that a given functor does not have a left or right adjoint.

First, consider the inclusion functor $I : \mathbf{Ring} \rightarrow \mathbf{Rng}$. If I has a right adjoint, then I is a left adjoint and needs to preserve colimits, and in particular, initial objects. But the initial object of **Ring** is \mathbb{Z} , while the initial object of **Rng** is $\{0\}$. Hence I does not have a right adjoint. One may show that it has a left adjoint given by the unitary (Dorroh) extension of a ring.

Now consider the inclusion functor $I : \mathcal{T} \rightarrow \mathbf{Ab}$. We have seen that it has the torsion functor as a right adjoint. If I has a left adjoint, then it is a right adjoint and needs to preserve limits, and in particular, products. But the product in \mathcal{T} of the family of torsion abelian groups $(\mathbb{Z}_p)_{p \text{ prime}}$ is $t\left(\prod_{p \text{ prime}} \mathbb{Z}_p\right) = \bigoplus_{p \text{ prime}} \mathbb{Z}_p$, while its product in **Ab** is their direct product. Hence T does not have a left adjoint.

(2) There are functors without any left or right adjoint. For instance take the functor $F : \mathbf{0} \rightarrow \mathcal{A}$ from the zero category to a non-zero category \mathcal{A} .

Also, there are functors having both a left and a right adjoint. E.g., see the example with the logical quantifiers. Alternatively, consider the following functors:

- The functor $L : \mathbf{Set} \rightarrow \mathbf{Top}$ defined by $L(X) = (X, \mathcal{P}(X))$ (discrete topology), and $L(f) = f : (X, \mathcal{P}(X)) \rightarrow (Y, \mathcal{P}(Y))$ for every function $f : X \rightarrow Y$.
- The forgetful functor $U : \mathbf{Top} \rightarrow \mathbf{Set}$.
- The functor $R : \mathbf{Set} \rightarrow \mathbf{Top}$ defined by $R(X) = (X, \{\emptyset, X\})$ (indiscrete topology), and $R(f) = f : (X, \{\emptyset, X\}) \rightarrow (Y, \{\emptyset, Y\})$ for every function $f : X \rightarrow Y$.

Then (L, U) and (U, R) are adjunctions.

Remark 7.9. There are some general results, such as Freyd's Adjoint Functor Theorem, on the existence of left or right adjoints.

Chapter 8

Additive Categories

8.1 Preadditive Categories

Definition 8.1. A category \mathcal{C} is called **preadditive** if $\text{Hom}_{\mathcal{C}}(A, B)$ is an additive abelian group for every objects A, B of \mathcal{C} , and the composition of morphisms is biadditive, i.e.,

$$(f_1 + f_2) \circ g = (f_1 \circ g) + (f_2 \circ g), \quad f \circ (g_1 + g_2) = (f \circ g_1) + (f \circ g_2)$$

for every suitable morphisms.

We denote by $0_{XY} : X \rightarrow Y$ the zero element of the abelian group $\text{Hom}_{\mathcal{C}}(X, Y)$.

Lemma 8.1. Let $f : W \rightarrow X$ and $g : Y \rightarrow Z$ be morphisms of a preadditive category \mathcal{C} . Then $0_{XY} \circ f = 0_{WY}$ and $g \circ 0_{XY} = 0_{XZ}$.

Proof. In order to show that an element of an abelian group is the zero element, it suffices to establish that its sum with itself is itself. Using the biadditivity of the composition, we have:

$$\begin{aligned} (0_{XY} \circ f) + (0_{XY} \circ f) &= (0_{XY} + 0_{XY}) \circ f = 0_{XY} \circ f, \\ (g \circ 0_{XY}) + (g \circ 0_{XY}) &= g \circ (0_{XY} + 0_{XY}) = g \circ 0_{XY}, \end{aligned}$$

which show the conclusion. □

Proposition 8.2. The following statements are equivalent for an object Y of a preadditive category \mathcal{C} :

- (1) Y is a zero object.
- (2) Y is initial.
- (3) Y is terminal.
- (4) $\text{Hom}_{\mathcal{C}}(Y, Y)$ is the zero group.
- (5) $0_{YY} = 1_Y$.

Proof. (1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (5) and (1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) These follow from the definition of everything involved: a zero object is by definition an initial and terminal object, initial and terminal objects by definition have a single endomorphism, and any two morphisms of the zero group are equal.

(5) \Rightarrow (1) Let X and Z be any objects of \mathcal{C} . For every arrows $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we have $f = 1_Y \circ f = 0_{YY} \circ f = 0_{XY}$ and $g = g \circ 1_Y = g \circ 0_{YY} = 0_{YZ}$ by Lemma 8.1. These imply that the morphism sets $\text{Hom}_{\mathcal{C}}(X, Y)$ and $\text{Hom}_{\mathcal{C}}(Y, Z)$ consist of precisely one morphism each. Because X and Z were arbitrary, we may conclude that Y is a zero object. □

Example 8.1. (1) As a first but not very typical example of a preadditive category, we consider any unitary ring R . We may thus consider the category \mathcal{C}_R induced by the multiplicative monoid of the ring - in other words, the category with a single object R , and the ring elements as morphisms with

composition given by ring multiplication. Endow the only morphism set $\text{Hom}_{\mathcal{C}_R}(R, R) = R$ with the additive group structure of the ring. That is, addition of morphisms is done by adding them as ring elements. Note now that this indeed defines a preadditive category: the only thing to verify is that the composition is biadditive, but this is exactly the requirement that multiplication distributes over addition in the ring, which is one of the ring axioms. One may also note the converse, that any preadditive category with a single object gives rise to a ring of morphisms with addition given by morphism addition and multiplication given by composition. The converse generalizes to categories with more than one object: in any preadditive category \mathcal{C} , every endomorphism set is a unitary ring with the ring addition given by morphism addition and the ring multiplication given by morphism composition. Two takeaways from this are that unitary rings may be thought of as precisely preadditive categories with a single object and that preadditive categories generalize unitary rings in the same way that arbitrary categories generalize monoids.

(2) We may by Proposition 8.2 conclude that there is no preadditive structure that **Set** can be endowed with to make a preadditive category, because it has initial objects that are not terminal and viceversa. Alternatively, note that there are empty morphism sets in **Set**, namely $\text{Hom}_{\mathbf{Set}}(X, \emptyset)$ for any non-empty set X , while the morphism sets are non-empty in preadditive categories, because they are groups.

(3) **Mon**, **Ring**, **Pos** and **Top** are not preadditive, because they do not have a zero object. One may also show that **Grp** is not preadditive.

Proposition 8.3. Let $f : X \rightarrow Y$ be a morphism in a preadditive category. Then f is a monomorphism if and only if $0 : 0 \rightarrow X$ is a kernel of f .

Dually, f is an epimorphism if and only if $0 : Y \rightarrow 0$ is a cokernel of f .

Proof. Assume first that f is a monomorphism. Clearly, $f \circ 0 = 0$. Next let $l : L \rightarrow X$ be a morphism such that $f \circ l = 0$. Then $f \circ l = f \circ 0$, which implies that $l = 0$, because f is a monomorphism.

$$\begin{array}{ccc} 0 & \xrightarrow{0} & X & \xrightarrow{f} & Y \\ \uparrow g & \nearrow l & & & \\ L & & & & \end{array}$$

Then there is a unique morphism $g : L \rightarrow 0$, namely $g = 0$, such that $0 \circ g = l = 0$. Hence $0 : 0 \rightarrow X$ is a kernel of f .

Conversely, assume that $0 : 0 \rightarrow X$ is a kernel of f . Let $l_1, l_2 : L \rightarrow X$ be morphisms such that $f \circ l_1 = f \circ l_2$. Then $f \circ (l_1 - l_2) = 0$.

$$\begin{array}{ccc} 0 & \xrightarrow{0} & X & \xrightarrow{f} & Y \\ \uparrow g & \nearrow l_1 - l_2 & & & \\ L & & & & \end{array}$$

Since $0 : 0 \rightarrow X$ is a kernel of f , there is a unique morphism $g : L \rightarrow 0$ such that $0 \circ g = l_1 - l_2$. Hence $l_1 = l_2$, which shows that f is a monomorphism.

The statement on epimorphisms follows by duality. □

Proposition 8.4. In a preadditive category, the Hom functors may be viewed as functors into the category of all abelian groups rather than just sets. In other words, for any preadditive category \mathcal{C} and fixed object A of \mathcal{C} , we have $\text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathbf{Ab}$ and $\text{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \rightarrow \mathbf{Ab}$.

Proof. The morphism sets in a preadditive category are abelian groups, so the mapping of the objects is sound. It remains to be shown that morphisms in \mathcal{C} are mapped to morphisms in **Ab**. For any morphism $f : X \rightarrow Y$, consider the function $\text{Hom}_{\mathcal{C}}(A, f) : \text{Hom}_{\mathcal{C}}(A, X) \rightarrow \text{Hom}_{\mathcal{C}}(A, Y)$ and verify that it is additive. For any morphisms $g, h \in \text{Hom}_{\mathcal{C}}(A, X)$ we have:

$$[\text{Hom}_{\mathcal{C}}(A, f)](g + h) = f \circ (g + h) = (f \circ g) + (f \circ h) = [\text{Hom}_{\mathcal{C}}(A, f)](g) + [\text{Hom}_{\mathcal{C}}(A, f)](h).$$

Similarly, we get for the contravariant Hom functor a function $\text{Hom}_{\mathcal{C}}(f, A) : \text{Hom}_{\mathcal{C}}(Y, A) \rightarrow \text{Hom}_{\mathcal{C}}(X, A)$, and for any morphisms $g, h \in \text{Hom}_{\mathcal{C}}(Y, A)$ we find that:

$$[\text{Hom}_{\mathcal{C}}(f, A)](g + h) = (g + h) \circ f = (g \circ f) + (h \circ f) = [\text{Hom}_{\mathcal{C}}(f, A)](g) + [\text{Hom}_{\mathcal{C}}(f, A)](h).$$

Hence every morphism in \mathcal{C} is mapped to a suitable morphism in \mathbf{Ab} . \square

8.2 Additive Categories

Definition 8.2. An **additive category** is a preadditive category with a zero object in which every pair of objects has a product.

Remark 8.5. The requirement that there exists a zero object may equivalently be viewed as a requirement that there are empty products. By this observation, we may equivalently define an additive category as a preadditive category in which every finite collection of objects has a product. Later on, we shall see that “product” may be replaced by “coproduct” and even “biproduct” in the definition of an additive category, which will explain the apparent asymmetry of the definition.

Example 8.2. Let us show that \mathbf{Ab} is an additive category.

The zero group $\{0\}$ is a zero object of \mathbf{Ab} . Any two objects of \mathbf{Ab} have a product, namely the direct product of the groups with the canonical projections.

Claim 8.6. $\text{Hom}_{\mathbf{Ab}}(M, N)$ is an abelian group for any abelian groups M and N .

All the necessary properties for the morphisms are inherited from the abelian group structure on N . The addition is well-defined in the sense that a sum of two morphisms is a morphism:

$$(f + g)(m + n) = f(m + n) + g(m + n) = f(m) + f(n) + g(m) + g(n) = f(m + n) + g(m + n)$$

for any $m, n \in M$. The remaining properties are verified in a similar fashion using the fact that the zero morphism and valewise subtraction of a morphism are both morphisms.

Claim 8.7. Composition of morphisms is biadditive in \mathbf{Ab} .

We need to prove that

$$(f_1 + f_2) \circ (g_1 + g_2) = (f_1 \circ g_1) + (f_1 \circ g_2) + (f_2 \circ g_1) + (f_2 \circ g_2)$$

holds for any suitable morphisms f_1, f_2, g_1 and g_2 in \mathbf{Ab} . For any m in the domain of g_1 and g_2 , we have:

$$\begin{aligned} [(f_1 + f_2) \circ (g_1 + g_2)](m) &= (f_1 + f_2)[(g_1 + g_2)(m)] = (f_1 + f_2)[g_1(m) + g_2(m)] \\ &= f_1[g_1(m) + g_2(m)] + f_2[g_1(m) + g_2(m)] \\ &= f_1[g_1(m)] + f_1[g_2(m)] + f_2[g_1(m)] + f_2[g_2(m)] \\ &= (f_1 \circ g_1)(m) + (f_1 \circ g_2)(m) + (f_2 \circ g_1)(m) + (f_2 \circ g_2)(m). \end{aligned}$$

Now the proposition follows from the claims and the existence of zero objects and products.

Example 8.3. (1) Additive categories: $\mathbf{Vect}(K)$, the category \mathcal{T} of torsion abelian groups.

(2) Any unitary ring R viewed as a category with only one object is a preadditive category, which is not additive, because it does not have a zero object.

(3) Non-additive categories: **Set**, **Mon**, **Ring**, **Pos**, **Top** (they do not have a zero object), **Grp** (finite products and coproducts do not coincide - see Proposition 8.8).

Proposition 8.8. Let X_1, X_2, \dots, X_n be objects of some preadditive category \mathcal{C} , which have a product $X = \prod_{j=1}^n X_j$ together with $\pi_j : X \rightarrow X_j$ (for $j = 1, \dots, n$). Set (for $i = 1, \dots, n$)

$\iota_i = (\delta_{ij})_{j=1}^n : X_i \rightarrow \prod_{j=1}^n X_j$, where $\delta_{ij} : X_i \rightarrow X_j$ is the morphism

$$\delta_{ij} = \begin{cases} 1_{X_i} & \text{if } i = j \\ 0_{X_i X_j} & \text{if } i \neq j \end{cases}$$

That is, ι_i is the morphism induced by a family consisting of a single identity morphism and the rest zero elements of the morphism groups. Then $\left(\prod_{j=1}^n X_j, (\iota_j)_{j=1}^n\right)$ is a coproduct of $(X_j)_{j=1}^n$.

Before proving the proposition, we will state and prove a useful result.

Proposition 8.9. $\iota_1 \circ \pi_1 + \cdots + \iota_n \circ \pi_n = 1_X$.

Proof. Consider the morphism $(\pi_1, \dots, \pi_n) : X \rightarrow X$, i.e., the unique morphism through which π_j (for $j = 1, \dots, n$) factors as $\pi_j = \pi_j \circ (\pi_1, \dots, \pi_n)$. On one hand, it is clearly the identity morphism 1_X for it has the required factorization property. On the other hand, we may show that it coincides with $\iota_1 \circ \pi_1 + \cdots + \iota_n \circ \pi_n$, because it too has the required factorization property: use the definition of all the ι_i in terms of the identity morphism and zero elements and the fact that zero elements compose to zero elements to get:

$$\begin{aligned} \pi_j \circ (\iota_1 \circ \pi_1 + \cdots + \iota_n \circ \pi_n) &= \pi_j \circ \iota_1 \circ \pi_1 + \cdots + \pi_j \circ \iota_n \circ \pi_n \\ &= \pi_j \circ \iota_1 \circ \pi_1 + \cdots + \pi_j \circ \iota_j \circ \pi_j + \cdots + \pi_j \circ \iota_n \circ \pi_n \\ &= 0_{X_i X} \circ \pi_1 + \cdots + 1_{X_j} \circ \pi_j + \cdots + 0_{X_n X_j} \circ \pi_n \\ &= 0_{X X_j} + \cdots + \pi_j + \cdots + 0_{X X_j} = \pi_j. \end{aligned}$$

This shows that $\iota_1 \circ \pi_1 + \cdots + \iota_n \circ \pi_n = 1_X$. \square

Now we continue to prove the Proposition 8.8.

Proof. Let $f_i : X_i \rightarrow Z$ (for $i = 1, \dots, n$) be a family of arbitrary morphisms into some object Z . We need to find a morphism $f : X \rightarrow Z$ through which all f_i factor via the proposed inclusion morphisms ι_i , and we need to prove that f is unique with this property. Consider $f = f_1 \circ \pi_1 + \cdots + f_n \circ \pi_n$. From the definition of the ι_i in terms of identity and zero elements, we see that f works: for any $j = 1, \dots, n$,

$$\begin{aligned} f \circ \iota_j &= (f_1 \circ \pi_1 + \cdots + f_n \circ \pi_n) \circ \iota_j = f_1 \circ \pi_1 \circ \iota_j + \cdots + f_n \circ \pi_n \circ \iota_j \\ &= f_1 \circ \pi_1 \circ \iota_j + \cdots + f_j \circ \pi_j \circ \iota_j + \cdots + f_n \circ \pi_n \circ \iota_j \\ &= f_1 \circ 0_{X_j X_1} + \cdots + f_j \circ 1_{X_j} + \cdots + f_n \circ 0_{X_j X_n} \\ &= 0_{X_j Z} + \cdots + f_j + \cdots + 0_{X_j Z} = f_j. \end{aligned}$$

To prove uniqueness of this f , let f' be any morphism with the same factorization property as f . By Proposition 8.9, we may rewrite f and f' and use that the f_i factor through them:

$$\begin{aligned} f &= f \circ 1_X = f \circ (\iota_1 \circ \pi_1 + \cdots + \iota_n \circ \pi_n) = f \circ \iota_1 \circ \pi_1 + \cdots + f \circ \iota_n \circ \pi_n \\ &= f_1 \circ \pi_1 + \cdots + f_n \circ \pi_n = f' \circ \iota_1 \circ \pi_1 + \cdots + f' \circ \iota_n \circ \pi_n \\ &= f' \circ (\iota_1 \circ \pi_1 + \cdots + \iota_n \circ \pi_n) = f' \circ 1_X = f'. \end{aligned}$$

Thus we have shown that $X = \prod_{j=1}^n X_j$ together with the ι_j (for $j = 1, \dots, n$) is a coproduct of the objects X_1, \dots, X_n . \square

Remark 8.10. As a consequence of the proposition, one may equivalently define additive categories to also have coproducts for any finite collection of objects. Thus, the dual of an additive category is an additive category (where the morphism addition in the dual category is understood to be the very same as in the original category).

8.3 Biproducts

Using Proposition 8.8 and its dual, we may conclude that coproducts and products coincide as objects in an additive category and may hence consider the objects equipped with both projection and inclusion morphisms. This notion of a simultaneous product and coproduct turns out to be useful in the study of additive categories and is captured in the following definition.

Definition 8.3. Let \mathcal{C} be a preadditive category and let X_1, \dots, X_n be a finite collection of objects in \mathcal{C} . A biproduct of X_1, \dots, X_n is an object X together with morphisms π_1, \dots, π_n (called projection morphisms) and ι_1, \dots, ι_n (called inclusion morphisms) satisfying for every $i, j \in \{1, \dots, n\}$: $\pi_j \circ \iota_i = \delta_{ij}$ and $\iota_1 \circ \pi_1 + \dots + \iota_n \circ \pi_n = 1_X$.

Remark 8.11. The notion of a biproduct is essentially self-dual in the sense that if X together with π_1, \dots, π_n and ι_1, \dots, ι_n is a biproduct of X_1, \dots, X_n in some preadditive category \mathcal{C} , then X together with ι_1, \dots, ι_n and π_1, \dots, π_n is a biproduct of the same objects X_1, \dots, X_n in \mathcal{C}^{op} (with the same morphism addition).

Remark 8.12. In terms of biproducts, Proposition 8.9 in the setting of Proposition 8.8 may be interpreted as saying that any product (or, by duality, coproduct) in a preadditive category may be expanded to a biproduct.

Remark 8.13. One may note that the definition of the biproduct of a family of objects conveniently refers only to these objects and the biproduct object (the definition is “internal”), whereas the definition of the product and coproduct in terms of their universal properties refers to virtually every object of the category (the definition is “external”).

By the very definition of the biproduct and in accordance with the previous remark, it is not obvious that every biproduct may be viewed as a simultaneous product and coproduct; the biproduct would seem to be more general than that. The following proposition shows, however, that biproducts indeed are simultaneous products and coproducts (with the obvious morphisms). Hence, finite products, finite coproducts, and biproducts coincide in preadditive categories.

Proposition 8.14. Let X together with π_1, \dots, π_n and ι_1, \dots, ι_n be a biproduct of some objects X_1, \dots, X_n in a preadditive category. Then:

- (1) $(X, (\pi_1, \dots, \pi_n))$ is a product of X_1, \dots, X_n ;
- (2) $(X, (\iota_1, \dots, \iota_n))$ is a coproduct of X_1, \dots, X_n .

Proof. In order to prove that $(X, (\iota_1, \dots, \iota_n))$ is a coproduct, we may proceed exactly as in the proof of Proposition 8.8, where the identities $\pi_j \circ \iota_i = \delta_{ij}$ and $\iota_1 \circ \pi_1 + \dots + \iota_n \circ \pi_n = 1_X$ were sufficient to prove the universal property of the coproduct (the former implied existence and the latter uniqueness of the factoring morphism). It follows by the above and that biproducts are “essentially” self-dual (Remark 8.11) that $(X, (\pi_1, \dots, \pi_n))$ is a coproduct in the opposite category and hence a product in the original category. \square

The following corollary to Propositions 8.8 and 8.14 summarizes the relationship between products, coproducts, and biproducts in a preadditive category.

Corollary 8.15. For a collection X_1, \dots, X_n of objects in a preadditive category, X_1, \dots, X_n have a product if and only if they have a coproduct if and only if they have a biproduct.

Example 8.4. Recall that the direct sum and the direct product of finitely many abelian groups M_1, \dots, M_n coincide. Interpreted categorically, we see that this is no coincidence: we know **Ab** to be additive (Example 8.2) and that the direct sum and direct product together with their inclusion and projection morphisms are a coproduct and, respectively, a product. By the uniqueness of products and coproducts and the above propositions, we may conclude that the direct sum and the direct product would at the very least have to be isomorphic. In fact, the direct sum and the direct product equipped with their canonical morphisms is a biproduct, which we may verify with ease. For any $i \in \{1, \dots, n\}$, consider the morphism $\pi_i \iota_i : M_i \rightarrow M_i$ and show that it is the identity morphism by noting that it fixes any element $m_i \in M_i$:

$$(\pi_i \iota_i)(m_i) = \pi_i((0, \dots, 0, m_i, 0, \dots, 0)) = m_i.$$

Also, $\pi_j \iota_i = 0_{M_i M_j}$ for every $j \neq i$. Next, consider

$$\iota_1 \pi_1 + \dots + \iota_n \pi_n : \bigoplus_{i=1}^n M_i \rightarrow \bigoplus_{i=1}^n M_i$$

and show that it is the identity morphism by noting that it too fixes any element, (m_1, \dots, m_n) say:

$$\begin{aligned} (\iota_1\pi_1 + \dots + \iota_n\pi_n)((m_1, \dots, m_n)) &= (\iota_1\pi_1)((m_1, \dots, m_n)) + \dots + (\iota_n\pi_n)((m_1, \dots, m_n)) \\ &= \iota_1(m_1) + \dots + \iota_n(m_n) \\ &= (m_1, 0, \dots, 0) + \dots + (0, \dots, 0, m_n) \\ &= (m_1, \dots, m_n). \end{aligned}$$

Proposition 8.16. If $(X, (\pi_i)_{i=1}^n, (\iota_i)_{i=1}^n)$ and $(X', (\pi'_i)_{i=1}^n, (\iota'_i)_{i=1}^n)$ are biproducts of the same family of objects X_1, \dots, X_n , then there is a unique isomorphism $\varphi : X \rightarrow X'$ that respects the projection and inclusion morphisms, i.e., with $\pi_i = \pi'_i \circ \varphi$ and $\iota'_i = \varphi \circ \iota_i$ for every i .

Proof. Proof. For uniqueness, we may view the biproducts as products (or coproducts) via Proposition 8.14 and use the uniqueness of products (or coproducts) up to isomorphism to conclude that there is precisely one isomorphism that respects the projection morphisms (or inclusion morphisms), and hence that there is at most one isomorphism that respects both projection and inclusion morphisms. We may also show the uniqueness more directly as follows:

$$\varphi = 1_{X'} \circ \varphi = \left(\sum_{i=1}^n \iota'_i \circ \pi'_i \right) \circ \varphi = \sum_{i=1}^n (\iota'_i \circ \pi'_i \circ \varphi) = \sum_{i=1}^n (\iota'_i \circ \pi_i).$$

Alternatively, we have:

$$\varphi = \varphi \circ 1_X = \varphi \circ \left(\sum_{i=1}^n \iota_i \circ \pi_i \right) = \sum_{i=1}^n (\varphi \circ \iota_i \circ \pi_i) = \sum_{i=1}^n (\iota'_i \circ \pi_i).$$

For existence, we may invoke Proposition 8.14 twice (once for the product and once for the coproduct). This yields two isomorphisms $\varphi, \varphi' : X \rightarrow X'$ with φ respecting the projection morphisms and φ' respecting the inclusion morphisms. By the chain of equalities above, $\varphi = \sum_{i=1}^n (\iota'_i \circ \pi_i) = \varphi'$. Hence the isomorphisms are the same and they respect both the projection morphisms and the inclusion morphisms. Another more explicit approach for the existence is to show that $\varphi = \sum_{i=1}^n (\iota'_i \circ \pi_i)$ respects the projection and inclusion morphisms and to show that the morphism $\psi = \sum_{i=1}^n (\iota_i \circ \pi'_i)$ is a two-sided inverse of φ . In any case, we have shown existence and uniqueness of an isomorphism between the biproduct objects that respects the projection and inclusion morphisms, which proves the proposition. \square

Remark 8.17. Much like for products and coproducts, the uniqueness of the biproduct object up to isomorphism guaranteed by Proposition 8.16 justifies the notation $\bigoplus_{i=1}^n X_i$ for the object of any biproduct of X_1, \dots, X_n . Somewhat sloppily, we will often use $\bigoplus_{i=1}^n X_i$ to refer to the entire biproduct (including the projection and inclusion morphisms).

8.4 Matrix Notation

The idea of viewing morphisms as matrices, with addition and composition of morphisms as matrix addition and multiplication, is fruitful not only in linear algebra (i.e., in the category $\mathbf{Vect}(K)$), but in any additive category.

Proposition 8.18. Let $X = \bigoplus_{j=1}^n X_j$ and $Y = \bigoplus_{i=1}^m Y_i$ be biproducts in an additive category \mathcal{A} .

A morphism $\varphi : X \rightarrow Y$ induces a family of morphisms $\{\varphi_{ij} \mid i = 1, \dots, m; j = 1, \dots, n\}$ with $\varphi_{ij} : X_j \rightarrow Y_i$ via $\varphi_{ij} = \pi_i \circ \varphi \circ \iota_j$. Conversely, any family of morphisms $\{\varphi_{ij} \mid i = 1, \dots, m; j = 1, \dots, n\}$ with $\varphi_{ij} : X_j \rightarrow Y_i$ induces a unique map $\varphi : X \rightarrow Y$ via the universal properties of the biproducts viewed as a coproduct and a product. The setting is depicted in the following diagram:

$$\begin{array}{ccc}
 \bigoplus_{j=1}^n X_j & \xrightarrow{\varphi} & \bigoplus_{i=1}^m Y_i \\
 \uparrow \iota_j & \searrow \varphi_i & \downarrow \pi_i \\
 X_j & \xrightarrow{\varphi_{ij}} & Y_i
 \end{array}$$

Given φ_{ij} , one gets φ_i by universal mapping property of the coproduct $\bigoplus_{j=1}^n X_j$, and then φ by universal mapping property of the product $\bigoplus_{i=1}^m Y_i$. For any morphism φ as above, arrange its corresponding morphisms into a matrix $[\varphi]$ as follows:

$$[\varphi] := \begin{bmatrix} \varphi_{11} & \cdots & \varphi_{1n} \\ \vdots & \ddots & \vdots \\ \varphi_{m1} & \cdots & \varphi_{mn} \end{bmatrix}$$

- (1) The correspondence is an isomorphism between the abelian groups $\text{Hom}_{\mathcal{A}}(X, Y)$ and $\bigoplus_{j=1}^n \bigoplus_{i=1}^m \text{Hom}_{\mathcal{A}}(X_j, Y_i)$.

In particular, the matrix of a sum is obtained by matrix addition:

$$[\varphi + \varphi'] = [\varphi] + [\varphi'],$$

where the addition of matrix entries in the right-hand side is just the morphism addition in the additive category.

- (2) Let $W = \bigoplus_{k=1}^p W_k$ be a third biproduct. If $\psi : W \rightarrow X$ and $\varphi : X \rightarrow Y$ are morphisms, then

$$(\varphi \circ \psi)_{ik} = \sum_{j=1}^n \varphi_{ij} \circ \psi_{jk}.$$

In other words, the matrix of the composition $\varphi \circ \psi$ is obtained by matrix multiplication

$$[\varphi \circ \psi] = [\varphi] \cdot [\psi],$$

where multiplication of the entries is done by composition and addition is the addition on morphisms in the additive category.

Pullbacks and pushouts exhibit a connection with kernels and cokernels in additive categories.

Proposition 8.19. Consider in an additive category a square and the following diagonal morphisms that it gives rise to:

$$\begin{array}{ccc}
 X & \xrightarrow{\alpha} & A \\
 \downarrow \beta & \searrow [\alpha] & \downarrow \varphi \\
 & A \oplus B & \\
 & \searrow [\varphi - \psi] & \\
 B & \xrightarrow{\psi} & Y
 \end{array}$$

Then the square is:

- (1) a pullback if and only if $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is a kernel of $[\varphi - \psi]$.
- (2) a pushout if and only if $[\varphi - \psi]$ is a cokernel of $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

Proof. We only prove the first statement, the other one following by duality.

Assume first that the square is a pullback. Then $[\varphi \ -\ \psi] \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \varphi \circ \alpha - \psi \circ \beta = 0$. Next let $\begin{bmatrix} \gamma \\ \delta \end{bmatrix} : X' \rightarrow A \oplus B$ be such that $[\varphi \ -\ \psi] \cdot \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = 0$, that is, $\varphi \circ \gamma = \psi \circ \delta$. By the universal mapping property of the pullback, there is a unique morphism $\chi : X' \rightarrow X$ such that $\alpha \circ \chi = \gamma$ and $\beta \circ \chi = \delta$, that is, $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot [\chi] = \begin{bmatrix} \alpha \circ \chi \\ \beta \circ \chi \end{bmatrix} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$. Hence $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is a kernel of $[\varphi \ -\ \psi]$.

$$\begin{array}{ccc} X & \xrightarrow{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}} & A \oplus B \xrightarrow{[\varphi \ -\ \psi]} Y \\ \uparrow \text{\scriptsize } x & \nearrow & \\ X' & & \end{array}$$

Conversely, assume that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ is a kernel of $[\varphi \ -\ \psi]$. Then $\varphi \circ \alpha - \psi \circ \beta = [\varphi \ -\ \psi] \cdot \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = 0$, hence $\varphi \circ \alpha = \psi \circ \beta$. Next let $\gamma : X' \rightarrow A$ and $\delta : X' \rightarrow B$ be morphisms such that $\varphi \circ \gamma = \psi \circ \delta$. Then $[\varphi \ -\ \psi] \cdot \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \varphi \circ \gamma - \psi \circ \delta = 0$. By the universal mapping property of the kernel, there is a unique morphism $\chi : X' \rightarrow X$ such that $\begin{bmatrix} \alpha \\ \beta \end{bmatrix} \cdot [\chi] = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$, that is, $\alpha \circ \chi = \gamma$ and $\beta \circ \chi = \delta$. Hence the square is a pullback. \square

Remark 8.20. The placement of the minus sign in front of ψ in the matrices for the diagonal morphisms is arbitrary. As noted in the end of the proof, the minus could equally well be placed in front of β , and one readily shows in similar fashion that placing it in front of either α or φ instead yields an equivalent statement.

8.5 Additive Functors

Definition 8.4. Let \mathcal{C} and \mathcal{D} be preadditive categories. Then a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called **additive** if $F(f + g) = F(f) + F(g)$ for any suitable morphisms f and g .

Remark 8.21. (1) If $F : \mathcal{C} \rightarrow \mathcal{D}$ is an additive functor, then $F(0_{XY}) = 0_{XY}$ for every objects X, Y of \mathcal{C} , and $F(-f) = -F(f)$ for every morphism f in \mathcal{C} .

(2) The composition of two additive functors is again an additive functor.

Proposition 8.22. If $F : \mathcal{C} \rightarrow \mathcal{D}$ is an additive functor between preadditive categories and X is a zero object of \mathcal{C} , then $F(X)$ is a zero object of \mathcal{D} .

Proof. Using the characterization of zero objects from Proposition 8.2, we have $0_{XX} = 1_X$. Applying F , we have $0_{F(X)F(X)} = 1_{F(X)}$, which is precisely to say that $F(X)$ is a zero object (using the characterization again). \square

Proposition 8.23. The following are equivalent for a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ between additive categories:

- (1) F is additive.
- (2) F respects binary products. In other words, if $(X \amalg Y, (\pi_X, \pi_Y))$ is a product of $X, Y \in \mathcal{A}$, then $(F(X \amalg Y), (F(\pi_X), F(\pi_Y)))$ is a product of $F(X), F(Y) \in \mathcal{B}$.
- (3) F respects binary coproducts. In other words, if $(X \amalg Y, (\iota_X, \iota_Y))$ is a coproduct of $X, Y \in \mathcal{A}$, then $(F(X \amalg Y), (F(\iota_X), F(\iota_Y)))$ is a coproduct of $F(X), F(Y) \in \mathcal{B}$.
- (4) F respects binary biproducts. In other words, if $(X \oplus Y, (\pi_X, \pi_Y), (\iota_X, \iota_Y))$ is a biproduct of $X, Y \in \mathcal{A}$, then $(F(X \oplus Y), (F(\pi_X), F(\pi_Y)), (F(\iota_X), F(\iota_Y)))$ is a biproduct of $F(X), F(Y) \in \mathcal{B}$.

Proof. (1) \Rightarrow (4) This follows readily from the fact that every additive functor respects the ingredients of the biproduct definition: identity morphisms, zero elements of the morphism groups, and morphism composition and addition. Let us verify this by assuming that F is additive, applying it to a biproduct $(X \oplus Y, (\pi_X, \pi_Y), (\iota_X, \iota_Y))$ in \mathcal{A} , and seeing that a biproduct in \mathcal{B} is indeed obtained. We have:

$$\begin{aligned} F(\pi_X) \circ F(\iota_X) &= F(\pi_X \circ \iota_X) = F(1_X) = 1_{F(X)}, \\ F(\pi_Y) \circ F(\iota_X) &= F(\pi_Y \circ \iota_X) = F(0_{XY}) = 0_{F(X)F(Y)}. \end{aligned}$$

Finally, the sum of compositions in the opposite order is the identity on the biproduct object:

$$F(\iota_X)F(\pi_X) + F(\iota_Y)F(\pi_Y) = F(\iota_X\pi_X) + F(\iota_Y\pi_Y) = F(\iota_X\pi_X + \iota_Y\pi_Y) = F(1_{X \oplus Y}) = 1_{F(X \oplus Y)}.$$

Thus we have shown that an additive functor respects binary biproducts.

(4) \Rightarrow (2) and (4) \Rightarrow (3) These are immediate by Remark 8.12 and Proposition 8.16 stating that products and coproducts may be expanded to biproducts and that a biproduct may be viewed as a product and coproduct. Given a (binary) product or coproduct in \mathcal{A} , expand it to a biproduct, use the assumption that F respects biproducts to obtain a biproduct in \mathcal{B} , and view this biproduct as a product or coproduct in \mathcal{B} (which will be of the required form, i.e., with the appropriate projection or inclusion morphisms) to conclude that F respects products and coproducts.

(2) \Rightarrow (1) and (3) \Rightarrow (1) Let $f, g : A \rightarrow A'$ be morphisms in \mathcal{A} . Then $f + g$ is the composition of the following morphisms:

$$A \xrightarrow{\begin{bmatrix} 1 \\ 1 \end{bmatrix}} A \oplus A \xrightarrow{\begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix}} A' \oplus A' \xrightarrow{\begin{bmatrix} 1 & 1 \end{bmatrix}} A'$$

Since F preserves binary (co)products, it follows that:

$$\begin{aligned} F(f + g) &= F\left(\begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} f & 0 \\ 0 & g \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = [F(1) \ F(1)] \cdot \begin{bmatrix} F(f) & 0 \\ 0 & F(g) \end{bmatrix} \cdot \begin{bmatrix} F(1) \\ F(1) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} F(f) & 0 \\ 0 & F(g) \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = F(f) + f(g). \end{aligned}$$

Hence F is additive. \square

The Hom functors for an additive category, which by Proposition 8.4 may be viewed as functors into the additive category \mathbf{Ab} , are an important example of additive functors.

Proposition 8.24. Let \mathcal{C} be a preadditive category and A a fixed object of \mathcal{C} . Then the Hom functors $\text{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \rightarrow \mathbf{Ab}$ and $\text{Hom}_{\mathcal{C}}(-, A) : \mathcal{C} \rightarrow \mathbf{Ab}$ are additive.

Proof. Note first that the proposition makes sense, seeing as \mathbf{Ab} is additive by Example 8.2. The proposition is a rather immediate consequence of the definition of the Hom functors in terms of composition and the biadditivity of the composition in an additive category. \square

Example 8.5. By Propositions 8.23 and 8.24, the morphism set for two biproducts $X_1 \oplus X_2$ and $Y_1 \oplus Y_2$ of an additive category \mathcal{A} decomposes as

$$\begin{aligned} &\text{Hom}_{\mathcal{A}}(X_1 \oplus X_2, Y_1 \oplus Y_2) \\ &= \text{Hom}_{\mathcal{A}}(X_1 \oplus X_2, -)(Y_1 \oplus Y_2) \\ &\cong \text{Hom}_{\mathcal{A}}(X_1 \oplus X_2, -)(Y_1) \oplus \text{Hom}_{\mathcal{A}}(X_1 \oplus X_2, -)(Y_2) \\ &= \text{Hom}_{\mathcal{A}}(-, Y_1)(X_1 \oplus X_2) \oplus \text{Hom}_{\mathcal{A}}(-, Y_2)(X_1 \oplus X_2) \\ &\cong \text{Hom}_{\mathcal{A}}(-, Y_1)(X_1) \oplus \text{Hom}_{\mathcal{A}}(-, Y_1)(X_2) \oplus \text{Hom}_{\mathcal{A}}(-, Y_2)(X_1) \oplus \text{Hom}_{\mathcal{A}}(-, Y_2)(X_2) \\ &= \text{Hom}_{\mathcal{A}}(X_1, Y_1) \oplus \text{Hom}_{\mathcal{A}}(X_1, Y_2) \oplus \text{Hom}_{\mathcal{A}}(X_2, Y_1) \oplus \text{Hom}_{\mathcal{A}}(X_2, Y_2). \end{aligned}$$

References

- [1] F.W. Anderson and K.R. Fuller. *Rings and Categories of Modules*. Graduate Texts in Mathematics. Springer New York, 2012.
- [2] S. Awodey. *Category Theory*. Oxford Logic Guides. Ebsco Publishing, 2006.
- [3] T. Bühler. Exact categories. *Expositiones Mathematicae*, 28(1):1–69, 2010.
- [4] S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag, New York, 1971. Graduate Texts in Mathematics, Vol. 5.
- [5] B. Mitchell. *Theory of Categories*. Pure and applied mathematics : a series of monographs and textbooks. Academic Press, 1965.
- [6] C. Năstăsescu. *Inele, Module, Categorii*. Editura Academiei Republicii Socialiste România, 1976.
- [7] S. Pettersson. *Additive, Abelian, and Exact Categories*. Uppsala Universitet, 2016.
- [8] I. Purdea. *Tratat de algebră modernă*, volume II. Editura Academiei, 1982.
- [9] B.T. Stenström. *Rings of Quotients: An Introduction to Methods of Ring Theory*. Springer-Verlag, 1975.