Quantum Field Theory

Prof. Martin Beneke

Preface

Contents

Introduction: Why Quantum Field Theory?

Systems with infinitely many degrees of freedom,
e.g. the classical electromagnetic field $A^\mu(x)$

Quantum theory of a single particle,
e.g. the wave function of an electron, $\psi(x)$

quantisation

compatibility with
relativity principle

Quantum Field Theory
$\phi(x)$ field operator

High energies $E \gg m$
$\Rightarrow$ particle production (e.g. e^+e^- pairs)
$\Rightarrow$ infinitely many degrees of freedom
Relativistic Causality
events at space-like separation,
$(x-y)^2 < 0$, cannot influence each other

- According to present knowledge QFT is the conceptual framework for all microphysical processes.
- QFT is the most important achievement of theoretical physics in the second half of the 20th century.
- Indispensible for mordern particle physics but also for many condensed matter systems.

Although the methods developed in this course are general, it focuses on relativistic QFT (i.e. theories obeying Lorentz invariance). The combination of the quantum and relativity principles is particularly powerful in constraining consistent theories.

Basic Paradigm

$$\boxed{S = \int \mathrm{d}^4x \; \mathcal{L}(\phi_i; g_i)}$$ defines a theory/system

action

Lagrange density

Fields
$\phi_i(x)$
$x = (t, \vec{x})$

fundamental parameters
(fine structure constant, electron mass,...)

- Which fields exist?

  (Fields are the primary objects, not the particles.)
- Which symmetries exist?

  (Symmetries constrain the structure of $\mathcal{L}$.)

In relativistic QFT, one always requires that the action is invariant under Lorentz transformations and space-time translations. More precisely:

$\boxed{g = (\underbrace{\Lambda}_{\text{boost+ rotation}}, \underbrace{a}_{\text{translation}})}$ element of the Poincaré group $\boxed{x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} + a^{\mu}}$

The states of a quantum system also transform, i.e.

$\boxed{(\Lambda, a) \mapsto U(\Lambda, a) \text{ operator on the Hilbert space}}$ $\boxed{|\psi'\rangle = U(\Lambda, a)|\psi\rangle \text{ operators } \mathcal{O}' = U(\Lambda, a)\mathcal{O}U^{-1}(\Lambda, a)}$

Invariance means that $|\langle\psi'|\varphi'\rangle| = |\langle\psi|\varphi\rangle|$ and $U(\Lambda, a)$ must be a unitary operator (possibly up to a phase).

A field operator $\mathcal{O}(x)$ is called a scalar operator, if

$$\boxed{U(\Lambda, a)\mathcal{O}(x)U^{-1}(\Lambda, a) = \mathcal{O}(\Lambda x + a)}$$

$\Rightarrow$ S is invariant, if $\mathcal{L}$ is a scalar operator——this is meant by relativistic QFT.

The Standard Model of particle physics which describes all observations (except for the existence of dark matter) is based on an amazingly small number of fields and symmetry principles.

Fields

Q_i, U_i, D_i	left- and right-handed quarks ($i = 1, 2, 3$ for three generations)
L_i, E_i	left- and right-handed leptons, left-handed only neutrinos
ϕ	the Higgs field
$A_{U(1)}$, $A_{SU(2)}$, $A_{SU(3)}$	Photon, Gluon, W/Z-Boson $U(1)$ $SU(3)$ $SU(2)$
$g_{\mu\nu}$	the gravitational metric field. The is no evidence yet that this is a quantum field. But there is no reason why it shouldn't be ($\to$ gravitons).

Symmetries

$$\underbrace{SU(3)}_{\text{strong interaction}} \times \underbrace{SU(2) \times U(1)}_{\text{electromagnetic+weak interaction}} \quad \text{gauge-symmetry}$$

Poincaré symmetry——see above

This data determines the Lagrangian. QFT then provides the tools to compute everything——in principle, at least. Sometimes, the equations are too difficult to solve them, even numerically. But an amazing variety of phenomena can be computed——systematically and precisely!

Chapter 1

Basic Concepts of Quantum Field Theory

1.1 Lagrange Formalism for Fields and Canonical Quantization

Classical Mechanics

In classical mechanics, we define

$L(q_n, \dot{q}_n)$	Lagrange function [assume no explicit time dependence]
q_n, $n = 1, \dots, N$	generalized coordinates

Then we can write action in the following, which is the functional of orbits $q_n(t)$ with fixed starting and endpoints $q_n(t_1)$, $q_n(t_2)$:

$$\boxed{S[q_n] = \int_{t_1}^{t_2} \mathrm{d}t \; L(q_n(t), \dot{q}_n(t))}$$

Equation of motion follows from the "action principle". The orbit taken by the physical system is the one where the action is stationary:

$$\boxed{\delta S[q_n] = 0}$$

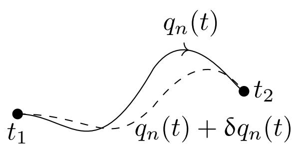
Or $\dfrac{\delta S}{\delta q_n(t)} = 0$ as functional derivative. This implies the Euler-Lagrange equations

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_n} - \frac{\partial L}{\partial q_n} = 0, \; \forall n}$$

Reminder:

$$\begin{aligned}
\delta S &= S[q_n + \delta q_n] - S[q_n] \\
&= \int_{t_1}^{t_2} \mathrm{d}t \left(\frac{\partial L}{\partial q_n}\delta q_n + \frac{\partial L}{\partial \dot{q}_n}\delta \dot{q}_n \right) \\
&= \underbrace{\left[\frac{\partial L}{\partial \dot{q}_n}\delta q_n \right]_{t_1}^{t_2}}_{\substack{= 0 \\ \text{because } \delta q_n(t_1) = \delta q_n(t_2) \equiv 0}} + \int_{t_1}^{t_2} \mathrm{d}t \underbrace{\left(\frac{\partial L}{\partial q_n} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_n} \right)}_{\frac{\delta S}{\delta q_n}} \delta q_n \stackrel{!}{=} 0
\end{aligned}$$

where δq_n is an infinitesimal functions see as the following figure.



And we use the summation convention

$$\delta\dot{q}_n \equiv \frac{\mathrm{d}}{\mathrm{d}t}\delta q_n$$

For arbitrary δq_n, this implies the Euler-Lagrange equation.

Transition to the Hamiltonian: We define the canonically conjugate momenta

$$\boxed{p_n \equiv \frac{\partial L}{\partial \dot{q}_n}} \tag{1.1}$$

then

$$\boxed{H(q_n, p_n) \equiv \sum_n p_n \dot{q}_n - L(q_n, \dot{q}_n)}$$

where $\dot{q}_n = \dot{q}_n(q_n, p_n)$ by solving (1.1) for $\dot{q}_n$.

Remark. Sometimes the system of equation (1.1) is degenerate and one cannot solve for all $\dot{q}_n$. This happens, for instance, if L does not depend on some of the $\dot{q}_n$, i.e. $L = L(q_n, \dot{q}_n, \widehat{q}_i)$, such that $\widehat{p}_i = 0$. In this case only the (q_n, p_n) but not the $(\widehat{q}_i, 0)$ are used as coordinates and H is defined as

$$H(q_n, p_n) = \sum_n p_n \dot{q}_n - L(q_n, \dot{q}_n, \widehat{q}_i).$$

The canonical coordinates fulfill the equations

$$\boxed{\{q_n, p_m\}_p = \delta_{nm}} \qquad \{q_n, q_m\} = \{p_n, p_m\}_p = 0$$

Then we have

$$\begin{aligned} \dot{q}_n &= \{q_n, H\}_p = \frac{\partial H}{\partial p_n} \\ \dot{p}_n &= \{p_n, H\}_p = -\frac{\partial H}{\partial q_n} \end{aligned}$$

where we define Poisson bracket

$$\{A, B\}_p = \frac{\partial A}{\partial q_n}\frac{\partial B}{\partial p_n} - \frac{\partial A}{\partial p_n}\frac{\partial B}{\partial q_n}$$

Notice that we have used summation convention.

Canonical Quantization

$L(q_n, \dot{q}_n)$ given. Interpret q_n, $\dot{q}_n$, p_n as operators on a Hilbert space. Impose the commutation relations

$$\boxed{\begin{aligned} &[q_n, p_n] = \mathrm{i}\delta_{nm} \\ &[q_n, q_m] = [p_n, p_m] = 0 \end{aligned}} \qquad \text{(at a fixed time)}$$

The time evolution of the operators is given by the Euler-Lagrange equation or equivalently by the Heisenberg equation

$$\boxed{\dot{q}_n = \frac{1}{\mathrm{i}}[q_n, H]}$$

and preserves the canonical, equal-time commutation relations.

Note.

- simply $\boxed{\{\ ,\ \}_p \to \frac{1}{\mathrm{i}}[\ ,\ ]}$
- We follow conventions and set $\hbar \equiv 1$. Since $\hbar$ is a universal constant, this defines the unit of energy. We can restore standard units by re-instating $\hbar$ factors until standard units match.

Lagrange-formalism for Fields

Generic field operator $\phi_n(x)$, where $x = (t, \vec{x})$.

The canonical coordinate/degree of freedom of the system is the value of the field at every point in space, e.g. the value of the electromagnetic potential A^μ, the elongation of a string, ...

Suppose space were discrete with points $\vec{x}$ on a lattice. Then there is a set $\phi_{n,\vec{x}(t)}$ of canonical coordinates, and the previous discussion applies with substitutions

$$\begin{array}{rcccl} q_n(t) & \to & \phi_{n,\vec{x}}(t) & \to & \phi_n(t,\vec{x}) \equiv \phi_n(x) \\ \sum\limits_n & \to & \sum\limits_{n,\vec{x}} & \to & \sum\limits_n \int \mathrm{d}^3\vec{x} \\ \delta_{nm} & \to & \delta_{nm}\delta_{\vec{x}\vec{y}} & \to & \delta_{nm}\delta^{(3)}(\vec{x}-\vec{y}) \\ L(q_n(t), \dot{q}_{n(t)}) & \to & L(\phi_n(x), \dot{\phi}_n(x)) & & \end{array}$$

In relativistic QFT are assume that L is an integral over a Lagrangian density:

$$\boxed{L = \int \mathrm{d}^3\vec{x}\, \mathcal{L}(\phi_n(x), \vec{\nabla}\phi_n(x), \dot{\phi}_n(x))}$$

where $\phi_n(x)$ means that all at one space-point: local. The reason is that it is difficult to construct non-local Lagrangians, which are compatible with relativistic causality. For a Lagrangian density, the action is automatically invariant, if $\mathcal{L}$ is a Lorentz-scalar, as discussed above.

Euler-Lagrange Equation

$$\begin{aligned} \delta S &= \delta \int \underbrace{\mathrm{d}t\, \mathrm{d}^3\vec{x}}_{\mathrm{d}^4x}\, \mathcal{L}(\phi_n, \underbrace{\vec{\nabla}\phi_n, \dot{\phi}_n}_{\partial_\mu\phi_n}) \\ &= \int \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi_n}\delta\phi_n + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\delta(\partial_\mu\phi_n) \right) \end{aligned}$$

Integrate by parts and assume $\phi_n(x) \to 0$ for $|\vec{x}| \to \infty$

$$\Rightarrow \qquad \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} \right) - \frac{\partial\mathcal{L}}{\partial\phi_n} = 0$$

Canonically conjugated fields

$$\boxed{\Pi_n(x) \equiv \frac{\partial\mathcal{L}}{\partial(\partial_0\phi_n(x))}}$$

Hamilton density

$$H = \sum_m \int \mathrm{d}^3\vec{x}\, \Pi_n(x)\partial_0\phi_n(x) - \int \mathrm{d}^3\vec{x}\, \mathcal{L} \equiv \int \mathrm{d}^3\vec{x}\, \mathcal{H}(x)$$

$$\Rightarrow \qquad \mathcal{H}(x) = \sum_n \Pi_n\partial_0\phi_n - \mathcal{L}$$

Canonical Quantization

Impose the equal-time commutation relations

$$\boxed{\begin{array}{l} [\phi_n(t,\vec{x}), \Pi_m(t,\vec{y})] = \mathrm{i}\delta_{nm}\delta^{(3)}(\vec{x}-\vec{y}) \\ [\phi_n(t,\vec{x}), \phi_m(t,\vec{y})] = [\Pi_n(t,\vec{x}), \Pi_m(t,\vec{y})] = 0 \end{array}}$$

The Euler-Lagrange equation is equivalent to the Heisenberg equation $\dot{\phi}_n(t,\vec{x}) = \frac{1}{\mathrm{i}}[\phi_n(t,\vec{x}), H]$ for the field operator. The formal solution is

$$\boxed{\phi_n(t,\vec{x}) = \mathrm{e}^{\mathrm{i}Ht}\phi_n(0,\vec{x})\mathrm{e}^{-\mathrm{i}Ht}}$$

Since H does not depend explicitly on t.

Note.

- If the commutation relations are imposed at one time t_0, they are preserved at all times.

$$\begin{aligned}[\phi_n(t,\vec{x}), \Pi_m(t,\vec{y})] &= [\mathrm{e}^{\mathrm{i}H(t-t_0)}\phi_n(t_0,\vec{x})\mathrm{e}^{-\mathrm{i}H(t-t_0)}, \mathrm{e}^{\mathrm{i}H(t-t_0)}\Pi_m(t_0,\vec{y})\mathrm{e}^{-\mathrm{i}H(t-t_0)}] \\ &= \mathrm{e}^{\mathrm{i}H(t-t_0)}\underbrace{[\phi_n(t_0,\vec{x}), \Pi_m(t_0,\vec{y})]}_{\mathrm{i}\delta_{nm}\delta^{(3)}(\vec{x}-\vec{y})}\mathrm{e}^{-\mathrm{i}H(t-t_0)} \\ &= \mathrm{i}\delta_{nm}\delta^{(3)}(\vec{x},\vec{y})\end{aligned}$$

- Unless otherwise mentioned, one uses the Heisenberg picture in QFT, where the time-dependence is in the operators, not the states. This is natural, because the equation of motion is for the field operator, and the field operators are the primary objects.

1.2 Noether Theorem for Fields

The Noether theorem establishes a relation between symmetries and conserved quantities. The generalization from mechanics to fields is straightforward, keeping in mind $q_n(t) \to \phi_n(t,\vec{x})$. There is also no difference between classical and quantum field theory. $\phi_n(x)$ could be a classical field or a quantum field operator.

$$S[\phi_n] = \int \mathrm{d}^4x\, \mathcal{L}(\phi_n, \partial_\mu\phi_n)$$

A symmetry is a transformation of the fields, which leaves the action invariant. Consider continuous symmetries and the infinitesimal one

$$\boxed{\begin{array}{l}\phi'_n(x) = \phi_n(x) + \varepsilon F_n(\phi'_n(x), \partial_\mu\phi'_n(x)) + \mathcal{O}(\varepsilon^2) \\ S[\phi'_n] = S[\phi_n]\end{array}}$$

Example: translations

$$\phi'_n(x) = \phi_n(x+a) = \phi_n(x) + a^\mu\partial_\mu\phi_n(x) + \cdots \quad \xRightarrow{a=\varepsilon} \quad F_n = \partial_\mu\phi_n$$

Noether Theorem

For every continuous symmetry, there is a conserved current

$$\boxed{j^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n(x))}F_n(x) - K^\mu(x)} \qquad [\partial_\mu j^\mu(x) = 0]$$

$$\text{and charge } \boxed{Q \equiv \int \mathrm{d}^3\vec{x}\, j^0(x)} \qquad [\frac{\mathrm{d}Q}{\mathrm{d}t} = 0]$$

Derivation:

Since the action is invariant by assumption, $\mathcal{L}$ can only change by a total derivative, i.e.

$$\delta\mathcal{L} = \mathcal{L}(\phi', \partial_\mu\phi') - \mathcal{L}(\phi, \partial_\mu\phi) \equiv \varepsilon\partial_\mu K^\mu(x) \tag{1.2}$$

for some K^μ. (In particular, $K^\mu \equiv 0$ if $\mathcal{L}$ itself is invariant, not only S.) On the other hand,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_n}\delta\phi_n + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\delta(\partial_\mu\phi_n) = \varepsilon\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}F_n\right) \tag{1.3}$$

where $\delta\phi_n = \varepsilon F_n$ and $\delta(\partial_\mu\phi_n) = \varepsilon\partial_\mu F_n$. We have used Euler-Lagrange equation to eliminate $\frac{\partial\mathcal{L}}{\partial\phi_n}$ at the second equality. Taking (1.3) minus (1.2) gives $\partial_\mu j^\mu(x) = 0$ for $j^\mu(x)$ as defined above. Also

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int \mathrm{d}^3\vec{x}\,\partial^0 j^0(x) \xlongequal{\partial_\mu j^\mu=0} -\int \mathrm{d}^3\vec{x}\,\vec{\nabla}\cdot\vec{j}(x) = 0$$

where at the last equality, assuming as usual that the fields vanish as $|\vec{x}| \to \infty$.

Note. While the invariance of the action holds for arbitrary field configurations, the conservation of the current is only true for the fields satisfying the field equations, because the Euler-Lagrange equation was used to obtain (1.3).

1.3 The Quantized Scalar Field

In this script, we develop the formalism of QFT for the simplest field operator, the scalar field. We shall generalize this later to fields with more complicated transformations under the Poincaré symmetry.

Note. The concept of mass dimension
Set $\hbar = 1$, $c = 1$. Again: we can define the unit of length as the distance light travels in one unit of time.

$$\boxed{\text{Notation } [X] \equiv \text{mass dimension of quantity } X}$$

Then

$$[\text{mass}] \equiv 1 \text{ by definition}$$

$$[H] = [\text{energy}] \xlongequal{E=mc^2,\ c=1} [m] = 1$$

$$[x] = [\text{length}] \xlongequal{\Delta x \Delta p = \hbar = 1} [1/\text{momentum}] \xlongequal{p=mc} [\text{mass}^{-1}] = -1$$

$\Rightarrow$

$$[\mathrm{d}^3 x] = -3$$
$$[\mathcal{L}] = [\mathcal{H}] = +4$$
$$[S] = [\text{action}] = 0$$

Construction Principles for a Lagrangian

- Specify the fields the theory should contain and the symmetries under which it should be invariant
- Build $\mathcal{L}$ as a sum of products of the field,

$$\boxed{\mathcal{L} = \sum_i g_i \theta_i[\phi]}$$

  such that S is a Lorentz scalar and invariant under all other postulated symmetries. Include all such terms in $\mathcal{L}$ up to mass dimension 4 or a specified number > 4. (We will understand this only much later.)
- $\mathcal{L}$ must contain derivatives of ϕ, $\partial_\mu \phi$. Otherwise there would be no dynamics, i.e. $\Pi = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = 0$ and $\phi(\vec{x})$ is not a canonical variable or a physical degree of freedom.

The Real (hermitian) Scalar Field

$\partial^\mu \phi \partial_\mu \phi$ is the simplest Lorentz scalar, which contains $\partial_\mu \phi$

$$\mathcal{L} > \frac{1}{2} \partial^\mu \phi \partial_\mu \phi$$

which is arbitrary, canonical normalization of the kinetic term for a real field.
ϕ has mass dimension 1, since $[\partial_\mu] = \left[\dfrac{1}{x}\right] = 1$
The most general Lagrangian is of the form

$$\boxed{\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi)}$$

- A Lagrangian of this form describes the Higgs field/particle in the Standard Model.
- V, the potential term, does not depend on $\partial_\mu \phi$. One could add higher order kinetic terms such as $K_1(\partial_\mu \partial_\nu \phi)(\partial^\mu \partial^\nu \phi)$, $K_2(\partial^2 \phi)^2$, but this is admissible only if the theory is interpreted as an effective QFT. We will discuss this later and do not add such terms now.
- Usually $V(\phi)$ is a polynomial in ϕ:

$$\boxed{V(\phi) = \lambda_0 + \lambda_1 \phi + \frac{1}{2} m^2 \phi^2 + \frac{\lambda_3}{3!} \phi^3 + \frac{\lambda_4}{4!} \phi^4 + \cdots}$$

λ_0 is irrelevant for the dynamics, since it drops out in the Euler-Lagrange equation. λ_1 can be removed by the field redefinition $\phi = \phi' - \frac{\lambda_1}{m^2}$, and can be set to zero, too.

- $[m] = 1$, $[\lambda_3] = 1$, $[\lambda_4] = 0$.

The canonically conjugated field is

$$\Pi = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \dot{\phi}$$

The Euler-Lagrange equation reads $\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi\right)$

$$\boxed{\partial_\mu \partial^\mu \phi + \frac{\mathrm{d}V}{\mathrm{d}\phi} = 0}$$

or

$$(\partial^2 + m^2)\phi + \underbrace{\frac{\lambda_3}{2}\phi^2 + \frac{\lambda_4}{3!}\phi^3}_{\text{non-linear terms, (self-) interactions}} = 0$$

1.3.1 Free Field, Creation and Annihilation Operators

A field theory is called free, if $\mathcal{L}$ contains only bilinear field products. Here we have the Klein-Gordon equation linear field equation

$$\boxed{\mathcal{L}_0 = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \quad \Rightarrow \quad (\partial^2 + m^2)\phi = 0}$$

Free field theories can be solved exactly by a Fourier ansatz

$$\boxed{\phi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p_0}(\mathrm{e}^{-\mathrm{i}px} a(\vec{p}) + \mathrm{e}^{\mathrm{i}px} a^\dagger(\vec{p}))} \tag{1.4}$$

where $px = p^0 t - \vec{p}\cdot\vec{x}$. Note that the operators $a(\vec{p})$ are independent of time. $\phi(x)$ is a solution of the Klein-Gordon equation provided $p^\mu = (p^0, \vec{p})$ satisfies $p^2 = m^2$, that is $p^0 = \sqrt{m^2 + \vec{p}^2}$.

We write $\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}$ rather than $\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$, because the former is Lorentz-invariant as can be seen from

$$\boxed{\frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2E_p} = \frac{\mathrm{d}^4 p}{(2\pi)^4} 2\pi\delta(p^2 - m^2)\theta(p^0)}$$

where $E_p = \sqrt{m^2 + \vec{p}^2}$. So this convention is convenient.

In the following, we use the short-hand $a(p)$ but always remember that $a(p)$ is labeled by $\vec{p}$, not the four-vector $p = p^\mu$, since p^0 is fixed through $p^0 = E_p = \sqrt{m^2 + \vec{p}^2}$ in such $\mathrm{d}^3\vec{p}$ integrals.

$\phi(x)$, $\Pi(x)$ satisfy the canonical commutation relations. For this to be true $a(p)$ and its adjoint $a^\dagger(p)$ must satisfy

$$\boxed{\begin{array}{l} [a_p, a^\dagger(p')] = 2E_p (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}') \\ [a(p), a(p')] = [a^\dagger(p), a^\dagger(p')] = 0 \end{array}} \tag{1.5}$$

By analogy with the harmonic oscillator and for more important reasons, which will become apparent shortly. $a(p)$ $[a^\dagger(p)]$ is called annihilation [creation] operator.

Verify that with (1.5) the field commutation relations are indeed satisfied. We first compute

$$\boxed{[\phi(x), \phi(y)] \xlongequal{\text{insert (1.4), use (1.5)}} \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}(\mathrm{e}^{-\mathrm{i}p(x-y)} - \mathrm{e}^{\mathrm{i}p(x-y)}) \equiv \Delta(x-y)}$$

$$\Rightarrow \qquad [\phi(t, \vec{x}), \phi(t, \vec{y})] = \Delta(x-y)|_{y^0 = x^0} = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}(\mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} - \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})}) = 0$$

where at the last equality we substitute $\vec{p} \to -\vec{p}$ in the 2nd term.

$$[\phi(t,\vec{x}),\Pi(t,\vec{y})] = \frac{\partial}{\partial y^0}\Delta(x-y)\bigg|_{y^0-x^0} = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\frac{\mathrm{i}}{2}(\mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})} + \mathrm{e}^{-\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})}) = \mathrm{i}\delta^{(3)}(\vec{x},\vec{y})$$

$[\Pi,\Pi]_{y^0=x^0}$ vanishes by similar argument as $[\phi,\phi]_{y^0=x^0}$.

We can prove the more general statement $[\phi(x),\phi(y)] = 0$ whenever $(x-y)^2 < 0$, i.e. x, y are space-like separated.

Proof. Since $\Delta(x-y)$ is Lorentz-invariant (as evident from its integral representation), we can evaluate it in any convenient frame. Since $(x-y)^2 < 0$, we choose a frame where $x^0-y^0 = 0$ and $(x-y)^2 = -(\vec{x}-\vec{y})^2 < 0$. The statement then follows, because we have already computed the equal-time commutator. For later, note that

$$\boxed{\Delta_+(x-y) \equiv \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}\mathrm{e}^{-\mathrm{i}p(x-y)}}$$

does not vanish at space-like separation.

It follows from $[\phi(x),\phi(y)] = 0$ for $(x-y)^2 < 0$ that the same is true for the Hamiltonian density which in turn determines the quantum time evolution. This guarantees the important property of relativistic causality.

The Hamilton operator of the free theory is

$$H_0 = \int \mathrm{d}^3\vec{x}\,(\Pi(x)\dot{\phi}(x) - \mathcal{L}_0(x)) = \int \mathrm{d}^3\vec{x}\,\frac{1}{2}\left(\left(\frac{\partial\phi}{\partial t}\right)^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)$$

where $\Pi = \dot{\phi}$. Now we insert (1.4) twice to get $\int \mathrm{d}^3\vec{x}\,\mathrm{d}^3\vec{p}\,\mathrm{d}^3\vec{p}\,'$. $\mathrm{d}^3\vec{x}$ integral can be done and gives δ-function. The aa and $a^\dagger a^\dagger$ terms vanish by $p^2 = m^2$. Therefore

$$\begin{aligned} H_0 &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}\frac{1}{2}\frac{1}{2p^0}2(p^0)^2(a(p)a^\dagger(p) + a^\dagger(p)a(p)) \\ &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}p^0\,\underbrace{a^\dagger(p)a(p)}_{\text{number operator}} + \underbrace{\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}\frac{1}{2}(2\pi)^3\delta^{(3)}(0)}_{\text{formally infinite ``zero-point'' energy because infinitely many degrees of freedom}} \end{aligned}$$

The first term suggests an interpretation as the energy being stored in $n(p)$ quanta with momentum $\vec{p}$ and energy $p^0 = \sqrt{m^2+\vec{p}^2}$. We will make this more precise in the following. ☐

1.3.2 Quanta of the Scalar Field and Particle States

Up to now, we did not specify the Hilbert space on which the field operator acts. We also need to identify the particle states or quanta of the field. Although we think of elementary particles as localized, often point-like, objects, it is more natural to think of quanta as quanta of energy of a particle with definite momentum. For instance, a photon with momentum $\vec{p} = \hbar\vec{k}$ has energy $E_p = |\vec{p}|c$.

To construct the Hilbert space, we assume that there is a state-vector $|\Omega\rangle$ in the Hilbert space, which satisfies

$$\boxed{a(p)|\Omega\rangle = 0, \quad \forall\vec{p}}$$

This state always exists. The argument is the same as for the harmonic oscillator, since for every $\vec{p}$, $a(p)$ and $a^\dagger(p)$ satisfy the same commutation relations (up to $\vec{p}$ being continuous). $|\Omega\rangle$ is called the vacuum state. Now define

$$\boxed{|p\rangle = a^\dagger(p)|\Omega\rangle}$$

Up to now $\vec{p}$ is only the Fourier-conjugate variable to $\vec{x}$ in the definition (1.4) and carries no physical interpretation. We now prove that the state $|p\rangle$ has momentum $\vec{p}$ and energy $E_p = \sqrt{m^2+\vec{p}^2}$. This justifies the interpretation of $\vec{p}$ as momentum and of the state $|p\rangle$ as a single quantum of the field ϕ, or a particle excitation with momentum $\vec{p}$ and rest mass m.

To prove this, recall that there is a deep connection between translation invariance and momentum conservation. Momentum is the Noether charge of translation symmetry.

From 1.2,

$$j^\mu(x) = \frac{\partial \mathcal{L}}{\partial(\partial_\mu)\phi} F_n(x) - K^\mu(x)$$

where $\dfrac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi$ and $F_n(x) = \partial^\alpha \phi$. Also $\alpha =$ translation into direction α ($\alpha = 0$ time translation). Noether current j^μ for every α.

$$\mathcal{L}'(x) = \mathcal{L}(x+a) = \mathcal{L}(x) + a_\alpha \partial^\alpha \mathcal{L}(x) = a_\alpha \partial_\mu (g^{\alpha\mu}\mathcal{L}) + \mathcal{L}(x)$$

where a_α is translation and ∂_μ is total derivative.

$$\Rightarrow \qquad K^{\alpha\mu} = g^{\alpha\mu}\mathcal{L}$$

The conserved current is “energy momentum tensor”

$$\boxed{T^{\alpha\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\alpha \phi - g^{\alpha\mu}\mathcal{L}}$$

And the charge is ($\mu = 0$ integral)

$$p^\alpha = \int \mathrm{d}^3\vec{x}\, T^{\alpha 0} = \int \mathrm{d}^3\vec{x} \left(\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)}}_{\Pi} \partial^\alpha \phi - g^{\alpha 0}\mathcal{L} \right)$$

- $\alpha = 0$

$$p^0 = \int \mathrm{d}^3 x\, \underbrace{(\Pi\dot{\phi} - \mathcal{L})}_{\mathcal{H}} = H$$

  Time translation $\to$ conservation of energy

- $\alpha = i$

  where $\partial^i = -\partial_i \leftrightarrow -\dfrac{\partial}{\partial x^i} = -\nabla^i$

In terms of creation and annihilation operators, the momentum operator is given by (integrate $\mathrm{d}^3\vec{x}$ after using (1.4), as before for H_0).

$$\begin{aligned}
\vec{P} &= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}\, \frac{1}{2}\vec{p}\,(a(p)a(-p) + a^\dagger(p)a(p) + a(p)a^\dagger(p) + a^\dagger(p)a^\dagger(-p)) \\
&= \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}\, \vec{p}\, a^\dagger(p)a(p)
\end{aligned}$$

where $a(p)a(-p)$ and $a^\dagger(p)a^\dagger(-p)$ vanishes since they are odd in $\vec{p}$. And $a(p)a^\dagger(p) = a^\dagger(p)a(p) +$ commutator term but it vanishes since odd in $\vec{p}$ and there is no zero-point three-momentum.

Note the similarity with H_0.

Now

$$\begin{aligned}
\vec{P}|p\rangle &= \int \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3 2k^0}\, \vec{k}\, a^\dagger(k)\, \underbrace{a(k)\ a^\dagger(p)}_{a^\dagger(p)a(k) + 2k^0(2\pi)^3\delta^{(3)}(\vec{p}-\vec{k})} |\Omega\rangle \\
&= 0 + \int \mathrm{d}^3\vec{k}\, \delta^{(3)}(\vec{p}-\vec{k})\ \vec{k}\ a^\dagger(k)|\Omega\rangle \\
&= \vec{p}\, a^\dagger(p)|\Omega\rangle \\
&= \vec{p}|p\rangle
\end{aligned}$$

Hence $|\vec{p}\rangle$ is a momentum eigenstate with momentum $\vec{p}$ as claimed.

Also $H_0|p\rangle = E_p|p\rangle$.

Hence $|p\rangle$ describes a relativistic particle with mass m, momentum $\vec{p}$ and energy $E_p = \sqrt{m^2+\vec{p}^2}$.

We can define multi-particle states by acting with several $a^\dagger(p_n)$ or by constructing the tensor product of the single-particle Hilbert space. If $|p\rangle$ denotes the momentum eigenstate basis of the single-particle Hilbert space, the following two definitions of the basis states of the N-particle Hilbert space are equivalent.

- $|p_1 p_2 \cdots p_n\rangle = \left(\prod_i \frac{1}{\sqrt{n_i(p)!}}\right) a^\dagger(p_1)a^\dagger(p_2)\cdots a^\dagger(p_N)|\Omega\rangle.$

  where i runs over all different values of p in the set $\{p_1, p_2, \ldots, p_N\}$ and $n_i(p)$ is the number of times p appears in $\{p_1, p_2, \ldots, p_N\}$.

- $|p_1 p_2 \cdots p_N\rangle = \frac{1}{\sqrt{N!}}\left(\prod_i \frac{1}{\sqrt{n_i(p)!}}\right) \sum_{\sigma\in S_N} P_\sigma(|p_1\rangle \otimes |p_2\rangle \otimes \cdots \otimes |p_N\rangle)$

  where σ sum over all permutations of $\{p_1, p_2, \ldots, p_N\}$.

Note that in the first definition, the state is automatically symmetric under the interchange of any two particles. Hence, the scalar field describes bosons. Since at every point in space there is only one degree of freedom, these bosons must have spin zero.

The union of all N-particle Hilbert spaces (of totally symmetric spaces) is the Hilbert space on which the field operator acts

$$\boxed{F^S \equiv \bigoplus_{N=0}^{\infty} H_N^S} \qquad \text{Fock space}$$

where $H_N^S = \underbrace{H \otimes H \otimes H \otimes \cdots \otimes H}_{N\text{-times, symmetrized}}$, H = single particle Hilbert space and H_0 = 1-dim Hilbert space spanned by $|\Omega\rangle$.

QFT resolves the wave-particle dualism of early quantum theory. There is only one fundamental entity, the quantum field. The microscopic excitations of the field have natural interpretations as quanta or particles. However, there are situations/states for which the non-commuting nature of $a(p)$ and $a^\dagger(p)$, or ϕ with Π is a small effect. In this case, $\phi(x)$ behaves like a classical field and (1.4) describes a superposition of classical plane waves.

1.3.3 The Complex (Non-hermitian) Scalar Field

$\phi(x)$ is non-hermitian field operator (complex classical field) and could reduce this to the case of the real field by writing

$$\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + \mathrm{i}\phi_2(x))$$

where $\phi_1(x)$ and $\phi_2(x)$ are real. However, it is more instructive to work with the complex field directly and to consider $\phi(x)$ and $\phi^\dagger(x)$ as independent canonical coordinates. For instance, the Hamilton density is

$$\mathcal{H} = \Pi\dot{\phi} + \Pi^\dagger\dot{\phi}^\dagger - \mathcal{L}$$
$$(= \Pi_1\dot{\phi}_1 + \Pi_2\dot{\phi}_2 - \mathcal{L})$$

which is certainly required to make $\mathcal{H}$ real and H hermitian. General Lagrangian (under the same assumptions as for the real field):

$$\boxed{\mathcal{L} = \partial_\mu\phi^\dagger\partial^\mu\phi - V(\phi, \phi^\dagger)}$$

For a complex scalar field the canonical normalization of the kinetic term $\partial_\mu\phi^\dagger\partial^\mu\phi$ is 1, not 1/2——gives the standard expression for H_0, see below. And V must be real, so that $\mathcal{L}$ is.

$$\boxed{V(\phi,\phi^\dagger) = m^2\phi^\dagger\phi + \frac{\lambda}{4}(\phi^\dagger\phi)^2 + [n\phi^2 + \lambda_{31}\phi^3 + \lambda_{32}\phi^2\phi^\dagger + \lambda_{41}\phi^4 + \lambda_{42}\phi^3\phi^\dagger + \text{hermitian conjugate}]}$$

where m, λ must be real.

This describes a theory of two spin-zero particles with different masses and different interactions. The corresponding real scalar fields are linear combinations of ϕ_1 and ϕ_2.

The complex scalar field is usually discussed for the case when one requires that the Lagrangian is invariant under the $U(1)$ symmetry group comprised by the transformations

$$\boxed{\phi(x) \to \phi'(x) = \mathrm{e}^{-\mathrm{i}\alpha}\phi(x)}$$

where α is real. This excludes all terms in $[\cdots]$ in $V(\phi, \phi^\dagger)$.

Euler-Lagrange equation

$$\boxed{\partial^2\phi + \frac{\partial V}{\partial \phi^\dagger} = (\partial^2 + m^2)\phi + \frac{\lambda}{2}(\phi^\dagger\phi)\phi = 0}$$

Complex conjugate as equation of motion from derivative with respect to ϕ.

Hamiltonian

$$H = \int \mathrm{d}^3\vec{x} \Big\{ \underbrace{\frac{\partial\phi^\dagger}{\partial t}\frac{\partial\phi}{\partial t} + (\vec{\nabla}\phi^\dagger)(\vec{\nabla}\phi) + m^2\phi^\dagger\phi}_{H_0} + \frac{\lambda}{4}(\phi^\dagger\phi)^2 \Big\}$$

where $\Pi = \dot{\phi}^\dagger$.

Free Theory and Particle States

The free equation of motion is again the Klein-Gordon equation $(\partial^2 + m^2)\phi = 0$. Since ϕ is not hermitian, we can write down the general solution in the form

$$\boxed{\phi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p_0}\mathrm{e}^{-\mathrm{i}px}a(p)} \tag{1.6}$$

This is a solution for $p^0 = \sqrt{m^2 + \vec{p}^2}$ and if a, $a^\dagger$ satisfy the commutation relations for standard annihilation/creation operators (as for the real field) it is easy to check that ϕ, Π satisfy the canonical field commutation relations. Nevertheless the above solution is not correct, because it is not consistent with relativistic causality. [One sometimes reads that the scalar field must describe two degrees of freedom at every point when it is complex, given by ϕ_1- and ϕ_2-particles, hence the above ansatz cannot be right. This argument is incorrect, because it assumes that we must use a and $a^\dagger$ in the field expansion. But this requirement is not necessary for a general QFT.]

To see this note that $\mathcal{H}$ contains $\phi^\dagger$ and ϕ. But with (1.6)

$$[\phi^\dagger(x), \phi(y)] = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0}\mathrm{e}^{\mathrm{i}p\cdot(x-y)}$$

which does not vanish at space-like distances (Page 9) and hence $[\mathcal{H}(x), \mathcal{H}(y)] \neq 0$ at $(x-y)^2 < 0$.

We already know that causality is respected if the term $\mathrm{e}^{\mathrm{i}px}a^\dagger(p)$ is added. Since $\phi(x)$ now need not be hermitian, we introduce a second set of annihilation and creation operators $b(p)$, $b^\dagger(p)$ and write

$$\boxed{\phi(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}(\mathrm{e}^{-\mathrm{i}px}a(p) + \mathrm{e}^{+\mathrm{i}px}b^\dagger(p))}$$

Note that we might have added the second term with coefficient λ. But we already know from the form of $\Delta(x-y)$ (Page 9) that the a and $b^\dagger$ term must contribute with relative coefficient one, so $|\lambda| = 1$. If λ is a phase, it can be absorbed into a redefinition of b and the phase of the b-particle states.

$\Rightarrow$ Relativistic causality implies that the complex scalar field describes two particles with equal mass (since $p^0 = \sqrt{m^2 + \vec{p}^2}$ for the a- and b-particles).

The free Hamiltonian is

$$\boxed{H_0 = \int \frac{\mathrm{d}^3 p}{(2\pi)^3 2p^0}p^0(a^\dagger(p)a(p) + b^\dagger(p)b(p))}$$

where we are dropping the zero-point energy.

Conserved Current

The $U(1)$ symmetry of $\mathcal{L}$ implies a conserved Noether charge.
We consider the infinitesimal variation $\phi' = \mathrm{e}^{-\mathrm{i}\varepsilon}\phi(x)$, then the variations are:

$$\begin{aligned}\delta\phi &= -\mathrm{i}\varepsilon\phi = \varepsilon F_\phi \\ \delta\phi^\dagger &= +\mathrm{i}\varepsilon\phi^\dagger = \varepsilon F_{\phi^\dagger}\end{aligned}$$

We have the conserved current

$$\begin{aligned}j^\mu(x) &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}F_\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}F_{\phi^\dagger} \\ &= -\mathrm{i}(\phi\partial^\mu\phi^\dagger - \phi^\dagger\partial^\mu\phi)\end{aligned}$$

where $K^\mu = 0$, since $\mathcal{L}$ itself is invariant——see 1.2. The conserved charge is

$$\boxed{Q = \int \mathrm{d}^3\vec{x}\, j^0(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}(a^\dagger(p)a(p) - b^\dagger(p)b(p))}$$

Suppose we assign a-particles charge 1 and b-particles charge -1. Then the total charge is Q (since $a^\dagger a$, $b^\dagger b$ are number operators) and this charge is conserved. Moreover, a- and b-particles have the same mass and interactions. Thus the b-particles is what is called the anti-particles (same mass, opposite charge) of the a-particle. This is an example of a general fact: relativistic causality requires that for any particle with non-zero charge(s), there is an anti-particle.

1.4 Interacting Fields, the Interaction Picture and the Perturbation Expansion

If $\mathcal{L}(\phi, \partial_\mu\phi)$ contains not only bilinear terms, the field equation is non-linear

$$(\partial^2 + m^2)\phi = \text{terms with products of } \phi \text{ (or } \phi^\dagger)$$

Not exactly solvable.

A solution in terms of time-dependent $a(p,t)$ is not very useful, since the problem is shifted into the unknown time-dependence of $a(p,t)$. Alternatively, the time-dependence of the field operator is given by

$$\phi(t,\vec{x}) = \mathrm{e}^{\mathrm{i}Ht}\phi(0,\vec{x})\mathrm{e}^{-\mathrm{i}Ht}$$

or

$$\frac{\partial}{\partial t}\phi(t,\vec{x}) = \mathrm{i}[H, \phi(t,\vec{x})]$$

but again no general solution is known.

If the interaction is weak, one can solve the equation in a perturbation expansion. As known from time-dependent perturbation theory in quantum mechanics, it is convenient to use the interaction picture, in which the time-dependence due to H_0 is in the operators and the one from the interaction in the states. The difference here is only that we start from the Heisenberg picture, since the interacting fields are in the Heisenberg picture (the Fock space states are time-independent).

Transition to the Interaction Picture

- Fix a reference time (here: $t_0 = 0$) and define

$$\boxed{\phi_I(0,\vec{x}) \equiv \phi(0,\vec{x})}$$

  where $\phi_I(0,\vec{x})$ is field in the interaction picture and $\phi(0,\vec{x})$ is interacting field.

- At $t = 0$ split $\boxed{H = H_0 + H_{\text{int}} = H_{I_0} + H_{I_{\text{int}}}}$ (at $t = 0$, $H_0 = H_{I_0}$ because the fields coincide). H_0 must be defined such that it can be represented as

$$\int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}p^0(a^\dagger(p)a(p) + \text{anti-particle contribution for the complex field})$$

- Define

$$\boxed{\phi_I(t,\vec{x}) \equiv e^{iH_0 t}\phi(0,\vec{x})e^{-iH_0 t}}$$

  where H_0 at $t=0$, or H_{I_0} since H_{I_0} is time-independent in the interaction picture, but H_0 is not time-independent in the Heisenberg picture even if H is. Or

$$\frac{\partial \phi_I}{\partial t}(t,\vec{x}) = i[H_{I_0}, \phi_I(t,\vec{x})]$$

  where H_{I_0} is the free Hamiltonian.

- The time evolution of interaction picture states is given through the definition

$$|\psi_I\rangle(t) = e^{iH_0 t}\underbrace{e^{-iHt}|\psi\rangle}_{\text{time-dependent Schrödinger picture state}}$$

  where we take out free evolution $e^{iH_0 t}$ from Schrödinger state and $|\psi\rangle$ is the Heisenberg state. This guarantees that Heisenberg picture, interacting fields

$$\begin{aligned}\langle\varphi|\mathcal{O}(t)|\psi\rangle &= \langle\varphi|e^{iHt}\underbrace{\mathcal{O}(0)}_{\mathcal{O}_I(0)}e^{-iHt}|\psi\rangle \\ &= \langle\varphi|e^{iHt}e^{-iH_0 t}\mathcal{O}_I(t)e^{iH_0 t}e^{-iHt}|\psi\rangle \\ &= \langle\varphi_I(t)|\mathcal{O}_I(t)|\psi_I(t)\rangle\end{aligned}$$

- At $t=0$, $H_{\text{int}}(\phi,\Pi) = H_{I_{\text{int}}}(\phi_I,\Pi_I)$. To construct the explicit expression, one has to express it in terms of ϕ_I, $\dot{\phi}_I$. Since

$$\boxed{\Pi_I = \frac{\partial \mathcal{L}_{0I}}{\partial(\partial_0\phi_I)} \text{ or } \dot{\phi}_I = i[H_{I_0}, \phi_I]}$$

  the functional forms $\dot{\phi}_I = \dot{\phi}_I(\phi_I,\Pi_I)$ and $\dot{\phi} = \dot{\phi}(\phi,\Pi)$ may be different, if H_{int} contains time-derivatives of the field.

Since the interaction picture fields have the time evolution of free fields, $H_{I_{\text{int}}}(\phi_I,\Pi_I)$ can be expressed in terms of creation and annihilation operators. This makes it easy to calculate the matrix elements in Fock space.

Example 1.1.

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m\phi + \mathcal{L}_{\text{int}}(\phi)$$

$$\xRightarrow{\Pi=\dot{\phi}} \qquad \mathcal{H} = \underbrace{\frac{1}{2}(\Pi^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2)}_{\mathcal{H}_0}\underbrace{-\mathcal{L}_{\text{int}}(\phi)}_{\mathcal{H}_{\text{int}}}$$

Then

$$\dot{\phi}_I = i[H_{I_0},\phi_I] = \int d^3\vec{x}\; i[\mathcal{H}_{I_0}(t,\vec{y}), \phi_I(t,\vec{x})] = \Pi_I$$

where $\mathcal{H}_{I_0}(t,\vec{y}) = \frac{1}{2}\Pi_I^2(t,\vec{y})$. The last equality is by canonical commutation rules. Hence

$$H_{I_{\text{int}}} = -\int d^3x\; \mathcal{L}_{\text{int}}(\phi_I)$$

This result holds very often, but not always.

Perturbation Expansion

$$\mathcal{O}(t) = \mathrm{e}^{\mathrm{i}Ht}\mathrm{e}^{-\mathrm{i}H_0 t}\mathcal{O}_I(t)\underbrace{\mathrm{e}^{\mathrm{i}H_0 t}\mathrm{e}^{-\mathrm{i}Ht}}_{\equiv U_I(t)}$$

where $\mathcal{O}_I(t)$ is time-dependence known. Then

$$\begin{aligned}\mathrm{i}\frac{\partial U_I}{\partial t} &= -H_{I_0}U_I + \mathrm{e}^{\mathrm{i}H_{I_0}t}(H_{I_0} + H_{I_{\mathrm{int}}}(0))\mathrm{e}^{-\mathrm{i}Ht} \\ &= \mathrm{e}^{\mathrm{i}H_{I_0}t}H_{I_{\mathrm{int}}}(0)\mathrm{e}^{-\mathrm{i}H_{I_0}t}U_I \\ &= H_{I_{\mathrm{int}}}(t)U_I\end{aligned}$$

where $H_{I_0} + H_{I_{\mathrm{int}}}(0)$ equals H at $t = 0$ and we insert $\mathrm{e}^{-\mathrm{i}H_{I_0}t}\mathrm{e}^{+\mathrm{i}H_{I_0}t}$ after $(H_{I_0} + H_{I_{\mathrm{int}}}(0))$ at the second equality.

$$\xLeftrightarrow{U_I(0)=\mathbb{1}} \qquad \boxed{U_I(t) = \mathbb{1} - \mathrm{i}\int_0^t \mathrm{d}t'\, H_{I_{\mathrm{int}}}(t')U_I(t)}$$

In these two equation, $H_{I_{\mathrm{int}}}(t)$ is small by assumption, therefore we can solve by interaction. The interactive solution ("successive approximation") is "Dyson series"

$$\begin{aligned}U_I(t) = \mathbb{1} - \mathrm{i}\int_0^t \mathrm{d}t'\, H_{I_{\mathrm{int}}}(t') + (-\mathrm{i})^2\int_0^t \mathrm{d}t'\int_0^{t'} \mathrm{d}t'' H_{I_{\mathrm{int}}}(t')H_{I_{\mathrm{int}}}(t'') + \cdots \\ + (-\mathrm{i})^n\int_0^t \mathrm{d}t_1'\int_0^{t_1'} \mathrm{d}t_2'\cdots\int_0^{t_{n-1}'} \mathrm{d}t_n'\, H_{I_{\mathrm{int}}}(t_1')\cdots H_{I_{\mathrm{int}}}(t_n') + \cdots\end{aligned}$$

Note: always $t' > t''$, i.e. the operator with larger time is on the left. The term in the second line is nth order term. Time-ordered product $T(A(t_1)B(t_2)\cdots)$ = product of operators in brackets such that the operators are ordered from left to right by decreasing time-argument. Then

$$\begin{aligned}&\int_0^t \mathrm{d}t_1'\int_0^{t_1'} \mathrm{d}t_2'\, H_{I_{\mathrm{int}}}(t_1')H_{I_{\mathrm{int}}}(t_2') = \int_0^t \mathrm{d}t_1'\mathrm{d}t_2'\, H_{I_{\mathrm{int}}}(t_1')H_{I_{\mathrm{int}}}(t_2') \\ &= \frac{1}{2}\int_0^t \mathrm{d}t_1'\mathrm{d}t_2'\, T(H_{I_{\mathrm{int}}}(t_1')H_{I_{\mathrm{int}}}(t_2'))\int_0^t \mathrm{d}t_1'\int_0^{t_1'} \mathrm{d}t_2'\cdots\int_0^{t_{n-1}'} \mathrm{d}t_n'\, H_{I_{\mathrm{int}}}(t_1')\cdots H_{I_{\mathrm{int}}}(t_2') \\ &= \frac{1}{n!}\int_0^t \mathrm{d}t_1'\cdots\mathrm{d}t_n'\, T(H_{I_{\mathrm{int}}}(t_1')\cdots H_{I_{\mathrm{int}}}(t_n'))\end{aligned}$$

Hence can cast the solution for $U_I(t)$ into the form

$$\boxed{U_I(t) = T\exp\left(-\mathrm{i}\int_0^t \mathrm{d}t'\, H_{I_{\mathrm{int}}}(t')\right)} \quad \text{time-ordered exponential}$$

in practice this can be calculated (with increasing work) order by order in the weak (by assumption) interaction, giving the time-dependence of the operator $\mathcal{O}(t)$ of the interacting/Heisenberg fields. A common situation in particle physics is that one prepares a state of two free particles at $t_i \to -\infty$ and measures the momenta of a state of free particles at $t_f \to +\infty$ after the interaction ("scattering") has taken place. The probability amplitude for the state $|i\rangle$ to have evolved into $|f\rangle$ is

$$\boxed{S_{fi} \equiv \lim_{\substack{t_i \to -\infty \\ t_f \to +\infty}} \langle f|U_I(t_f, t_i)|i\rangle} \qquad \text{(all in the interaction picture)}$$

where $U_I(t_f, t_i)$ is time evolution operator in the interaction picture

$$\mathrm{e}^{\mathrm{i}H_0 t_f}\mathrm{e}^{-\mathrm{i}H(t_f - t_i)}\mathrm{e}^{-\mathrm{i}H_0 t_i} = U_I(t_f)U_I^{-1}(t_i)$$

And

$$\mathrm{i}\frac{\partial}{\partial t}U_I(t, t_i) = H_{I_{\mathrm{int}}}(t)U_I(t, t_i)$$

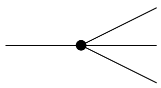
So

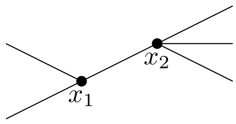
$$
\begin{aligned}
S_{fi} &= \langle f| T \exp\left(-\mathrm{i} \int_{-\infty}^{+\infty} \mathrm{d}t \, H_{I_{\text{int}}}(t)\right) |i\rangle \\
&= \langle f| T \exp\left(-\mathrm{i} \int \mathrm{d}^4x \, \mathcal{H}_{I_{\text{int}}}(x)\right) |i\rangle
\end{aligned}
$$

where we express $\langle f|$, $\mathcal{H}_{I_{\text{int}}}(x)$ and $|i\rangle$ in terms of annihilation and creation operators. For example if $|i\rangle$ consists of two particles and $|f\rangle$ of four and $\mathcal{H}_{I_{\text{int}}} = \frac{\lambda}{4!}\phi_I^4$ then

$$
\begin{aligned}
S_{fi} \sim \; & \langle \Omega | a(p_{f1}) \cdots a(p_{f4}) \\
& \left[\mathbb{1} - \mathrm{i} \int \mathrm{d}^4x \, \frac{\lambda}{4!}\phi_I^4(x) + \frac{(-\mathrm{i})^2}{2} \int \mathrm{d}^4x_1 \mathrm{d}^4x_2 \, T\left(\frac{\lambda}{4!}\phi_I^4(x_1) \frac{\lambda}{4!}\phi_I^4(x_2)\right) + \cdots \right] a^\dagger(p_{1i}) a^\dagger(p_{2i}) |\Omega\rangle
\end{aligned}
$$

pick $(a^\dagger)^3 a$ pick $a^\dagger a^\dagger a a$ $(a^\dagger)^3 a$





$$
\sim 0 + \lambda \times \underbrace{0}_{\substack{\text{vanishes, because assume} \\ p_{1i},\, p_{2i},\, p_{1f} \cdots p_{4f} \text{ are all different}}} + \lambda^2
$$

Using this formalism and quantum mechanical scattering theory, one can compute relativistic scattering of states which follow from QFT formulations of physical systems. In the following, however, a more powerful formalism shall be developed, which does not make use of the interaction picture.

Chapter 2

Path Integral Representation of Quantum Field Theory

Can be defined in single-particle quantum mechanics. Generalization to quantum field is rather straight-forward, as before, keeping in mind $q_n(t) \to \phi_{n,\vec{x}}(t)$.

The path integral was invented by Feynman in 1942, but was recognized as a very powerful tool in QFT only in the late 1960/1970s for quantizing non-abelian gauge theories and numerical simulations of strongly interacting QFTs (no perturbation expansion is possible) on a discrete space-time.

An advantage of the path integral representation is that Lorentz invariance is manifest, since the Hamiltonian is not explicitly needed, contrary to the canonical/operator formalism. On the other hand, unitarity of the time evolution is not manifest.

One may define QFTs through the path integral. We will not take this point of view here, since it is problematic in continuous space-time, where the path integral is not always mathematically well-defined non-perturbatively. Rather we derive the path integral representation from the canonical formalism, which also proves the equivalence of both approaches.

In this chapter, we consider first the so-called Green functions

$$\boxed{G(x_1, \ldots, x_n) \equiv \langle\Omega|T(\phi(x_1)\cdots\phi(x_n))|\Omega\rangle}$$

where $|\Omega\rangle$ is the ground state of the interacting theory. Not to be confused with the Fock space vacuum state (which we now call $|0\rangle$), which is related to the free Hamiltonian. We require that the ground state $\equiv$ vacuum state is Lorentz-invariant, i.e.

$$\boxed{U(\Lambda, a)|\Omega\rangle = |\Omega\rangle}$$

We will establish later the relation between Green functions and observables such as scattering matrix elements.

2.1 Path Integral Formula for Green Functions

2.1.1 Hamilton Version of the Path Integral

Let $Q_n(t)$, $P_n(t)$ be canonical coordinates (operators) and conjugate momenta $H(Q_n, P_n)$ Hamiltonian, respectively. Q_n, P_n are Hermitian operators (real variables) with canonical commutation relations

$$\boxed{\begin{aligned}&[Q_n(t), P_m(t)] = \mathrm{i}\delta_{nm}\\ &[Q_n(t), Q_m(t)] = [P_n(t), P_m(t)] = 0\end{aligned}}$$

Q_n could be $\{X_1, X_2, X_3\}$, the position operators of a single particle, or $\phi_{n,\vec{x}}$ the field values at every space point, in which case n in Q_n is a continuous index. We will use the discrete index notation first.

Since the Q_n and P_n commute among themselves, can find a simultaneous eigenstate for all Q_n's, or all P_n's:

$$\boxed{\begin{aligned}&Q_n(t)|q;t\rangle = q_n|q;t\rangle\\ &P_n(t)|p;t\rangle = p_n|p;t\rangle\end{aligned}}$$

We assume that the system is specified completely by eigenvalues of the canonical coordinates or conjugate momenta, hence the eigenstates can be taken to an orthonormal basis for every t:

$$\boxed{\begin{gathered}\langle q';t|q;t\rangle = \prod_n \delta(q'_n - q_n) \\ \prod_n \mathrm{d}q_n\,|q;t\rangle\langle q;t| = 1\end{gathered}}$$

Similarly for $|p;t\rangle$ states. The commutation relation implies

$$p_m\langle q;t|p;t\rangle = \rangle q;t|P_m|p;t\rangle = \frac{1}{\mathrm{i}}\frac{\partial}{\partial q_m}\langle q;t|p;t\rangle$$

where the second equality is the proof from commutation relations as in quantum mechanics.

$$\Rightarrow \qquad \boxed{\langle q;t|p;t\rangle = \prod_n \mathrm{e}^{\mathrm{i}q_n p_n}}$$

Since $Q_n(t)$, $P_n(t)$ are Heisenberg operators:

$$\begin{aligned} Q_n(t) &= \mathrm{e}^{\mathrm{i}Ht}Q_n(0)\mathrm{e}^{-\mathrm{i}Ht} \\ |q;t\rangle &= \mathrm{e}^{\mathrm{i}Ht}|q\rangle \text{ with } |q\rangle \equiv |q;t=0\rangle \end{aligned}$$

Similarly for $P_n(t)$; $|p;t\rangle$.

$$\Rightarrow \qquad \boxed{\langle q_f;t_f|p_i;t_i\rangle = \langle q_f|\mathrm{e}^{-\mathrm{i}H(t_f-t_i)}|q_i\rangle} \tag{2.1}$$

where $|q_f;t_f\rangle$, $|q_i,t_i\rangle$ are "position" eigenstate at time t_f, t_i. Probability amplitude that the system is in the coordinate eigenstate q_f at time t_f, if it was in eigenstate q_i at t_i.

Remark. H is a function of Q_n, P_m. In quantum theory the order matters. Here we adopt the convention that all Q's are to the left of all P's. If this is not the case, we can always achieve this by commuting the Q's to the left by applying the canonical commutation rules. The function obtained in this way is the called $H(Q_n, P_m)$ to be used in the following.

We now derive a path integral representation for the transition amplitude (2.1):

$$\langle q';t+\mathrm{d}t|q;t\rangle = \langle q'|\mathrm{e}^{-\mathrm{i}H\mathrm{d}t}|q\rangle = \int\prod_n \frac{\mathrm{d}p_n}{2\pi}\langle q'|\mathrm{e}^{-\mathrm{i}H(Q_n,P_n)\mathrm{d}t}|p\rangle\langle p|q\rangle$$

where $\mathrm{d}t$ is infinitesimal. Now use

$$\mathrm{e}^{-\mathrm{i}H(Q_n,P_n)\mathrm{d}t} \approx 1 - \mathrm{i}H(Q_n,P_n)\mathrm{d}t \xlongequal{\text{in } \langle q'|[\cdots]|p\rangle} 1 - \mathrm{i}H(q'_n,p_n)\mathrm{d}t \approx \mathrm{e}^{-\mathrm{i}H(q'_n,p_n)\mathrm{d}t}$$

This is a number. So

$$\boxed{\langle q';t+\mathrm{d}t|q;t\rangle = \int\prod_n \frac{\mathrm{d}p_n}{2\pi}\mathrm{e}^{-\mathrm{i}H(q'_n,p_n)\mathrm{d}t + \mathrm{i}\sum_n(q'_n - q_n)p_n}}$$

where $\mathrm{i}\sum_n(q'_n - q_n)p_n$ is from $\langle q'|p\rangle\langle p|q\rangle$. Split $[t_i, t_f]$ into $N+1$ small intervals

$$\begin{aligned} t_i \equiv t_0 < t_1 < \cdots < t_N < t_{N+1} \equiv t_f \\ \mathrm{d}t = t_{k+1} - t_k = \frac{t_f - t_i}{N+1} \end{aligned} \qquad \text{(equidistant)}$$

Insert complete sets of position eigenstates and use the above formula for the infinitesimal time interval

$$\begin{aligned}\langle q_f;t_f|q_i;t_i\rangle &= \int\prod_n \mathrm{d}q_{1n}\prod_n \mathrm{d}q_{2n}\cdots\prod_n d_{q_{Nn}}\langle q_f;t_f|q_N;t_n\rangle\langle q_N;t_N|q_{N-1};t_{N-1}\rangle\cdots\langle q_1;t_1|q_i;t_i\rangle \\ &= \int\prod_{k=1}^N\prod_n \mathrm{d}q_{kn}\prod_{k=0}^N\prod_n\frac{\mathrm{d}p_{kn}}{2\pi}\mathrm{e}^{\mathrm{i}\sum_{k=1}^{N+1}\left(\sum_n(q_{kn} - q_{k-1\,n}) - H(q_{kn},p_{k-1\,n})\mathrm{d}t\right)}\end{aligned}$$

where $q_0 = q_i$, $q_{N+1} = q_f$. Take $N \to \infty$, $\mathrm{d}t \to 0$.

$$\begin{aligned} q_{kn} &\to q_n(t) \\ p_{kn} &\to p_n(t) \\ q_{kn} - q_{k-1\,n} &\to \dot{q}_n(t)\mathrm{d}t \\ \sum_{k=1}^{N+1} \mathrm{d}t &\to \int_{t_i}^{t_f} \mathrm{d}t \end{aligned}$$

where $q_n(t)$ and $p_n(t)$ are functions of t.

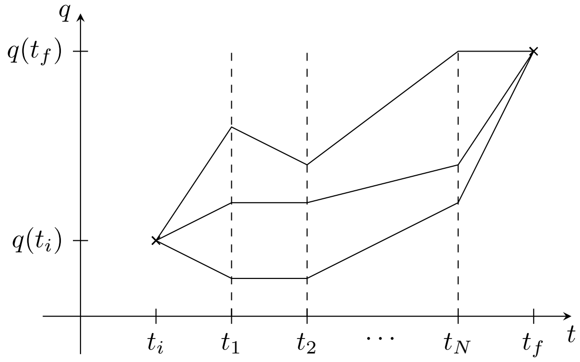
$$\int \prod_{k=1}^{N}\prod_{n} \mathrm{d}q_{kn} \prod_{k=0}^{N}\prod_{n} \frac{\mathrm{d}p_n}{2\pi} \to \int\limits_{\substack{q(t_i)\,=\,q_i \\ q(t_f)\,=\,q_f}} \prod_t \prod_n \mathrm{d}q_n(t) \prod_t \prod_n \frac{\mathrm{d}p_n(t)}{2\pi} \equiv \int\limits_{\substack{q(t_i)\,=\,q_i \\ q(t_f)\,=\,q_f}} \mathcal{D}[q_n(t)]\mathcal{D}[p_n(t)]$$

Integrate over all functions $q_n(t)$, $p_n(t)$ "paths" from $q_n(t_i)$ to $q_n(t_f)$. Hence

$$\boxed{\langle q_f; t_f | q_i; t_i \rangle = \int\limits_{\substack{q(t_i)\,=\,q_i \\ q(t_f)\,=\,q_f}} \mathcal{D}[q_n(t)]\mathcal{D}[p_n(t)] \mathrm{e}^{\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}t \left(\sum_n \dot{q}_n(t) p_n(t) - H(q_n(t), p_n(t)) \right)}} \tag{2.2}$$

Measure $\mathcal{D}$ is formally defined by going back to the discrete version.

At every t_k, one integrates over all q_k. This can be interpreted as integrating (summing) over all paths/trajectories from q_i to q_f with the (complex) weight given by the integrand of (2.2). Same for p_k but here one also integrates over the unconstrained boundary values p_i, p_f.



Generalization to fields:

$$\boxed{H = \int \mathrm{d}^3\vec{x}\, \mathcal{H}(\phi(x), \Pi(x))}$$

For simplicity, $\phi(x)$ is Hermitian scalar field.

$$\sum_n \to \int \mathrm{d}^3\vec{x}$$

$$|q;t\rangle \to |\phi;t\rangle = \text{simultaneous eigenstate of all "coordinates"}$$

$$\phi_{\vec{x}}(t),\ \text{i.e.}\ \boxed{\phi(t,\vec{x})|\phi;t\rangle = \phi(\vec{x})|\phi;t\rangle}$$

where $\phi(t,\vec{x})$ is field operator and $\phi(\vec{x})$ is function (a number for every $\vec{x}$).

$$\boxed{\langle \phi_f; t_f | \phi_i; t_i \rangle = \int\limits_{\substack{\phi(t_i,\vec{x})\,=\,\phi_i(\vec{x}) \\ \phi(t_f,\vec{x})\,=\,\phi_f(\vec{x})}} \mathcal{D}[\phi(x)]\mathcal{D}[\Pi(x)] \mathrm{e}^{\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}t \int \mathrm{d}^3\vec{x}\, (\Pi(x)\dot{\phi}(x) - \mathcal{H}(\phi(x), \Pi(x)))}}$$

The left hand side means the quantum state. The right hand side involves only c-number, classical field configurations $\phi(x)$, $\Pi(x)$.

The formula says that the quantum transition amplitude from a field configuration $\phi_i(\vec{x})$ at time t_i to the configuration $\phi_f(\vec{x})$ at t_f is given by the sum over all classical field configurations with the weight specified by the integrand.

2.1.2 Path Integral for Green Functions

Eventually we are interested in scattering amplitudes of particle states and Green functions, not of field configurations. In a first step, we generalize (2.2) to

$$\boxed{\langle q_f; t_f | \mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\cdots | q_i; t_i\rangle}$$

where $\mathcal{O}_a(t_a)$ and $\mathcal{O}_b(t_b)$ are arbitrary operators $\mathcal{O}(t) = \mathcal{O}(P_n(t), Q_n(t))$ with the convention that all Q's are to the right of all P's. As before divide $[t_i, t_f]$ into small intervals at t_k and insert complete sets of states $|q_k; t_k\rangle$ at t_k. This can be done only if the operators $\mathcal{O}_a(t_a)$, $\mathcal{O}_b(t_b)$ are time-ordered, i.e. $t_f > t_a > t_b > \cdots > t_i$. If this is not the case, we must put them in order. That is, we compute

$$\boxed{\langle q_f; t_f | T(\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\cdots) | q_i; t_i\rangle}$$

If t_a falls into $t \ldots t + \mathrm{d}t$, we have expressions of the form

$$\begin{aligned}\langle q'; t+\mathrm{d}t | \mathcal{O}_a(P_n(t_a), Q_n(t_a)) | q; t\rangle &= \int \prod_n \frac{\mathrm{d}p_n}{2\pi} \langle q' | \mathrm{e}^{-\mathrm{i}H(Q_n,P_n)\mathrm{d}t} | p\rangle \langle p | \mathcal{O}_a(P_n(t_a), Q_n(t_a)) | q\rangle \\ &\xlongequal[\text{for } \mathrm{d}t\to 0]{t_a \to t} \int \prod_n \frac{\mathrm{d}p_n}{2\pi} \mathrm{e}^{-\mathrm{i}H(q_n', p_n)\mathrm{d}t + \mathrm{i}\sum (q_n' - q_n)p_n} \mathcal{O}_a(p_n, q_n)\end{aligned}$$

It is easy to evaluate $\langle p | \mathcal{O}_a(P_n(t_a), Q_n(t_a)) | q\rangle$, since all P's to the left. $\mathcal{O}_a(p_n, q_n)$ is now a function of p_n, q_n, no longer an operator. The remaining manipulations are exactly as before.

$$\boxed{\begin{aligned}&\langle q_f; t_f | T(\mathcal{O}_a(t_a)\mathcal{O}_b(t_b)\cdots) | q_i; t_i\rangle \\ &= \int_{\substack{q(t_i) = q_i \\ q(t_f) = q_f}} \mathcal{D}[q_n(t)]\mathcal{D}[p_n(t)] \mathrm{e}^{\mathrm{i}\int_{t_i}^{t_f} \mathrm{d}t \left(\sum_n \dot{q}_n(t)p_n(t) - H(q_n(t), p_n(t))\right)} \mathcal{O}_a(P_n(t_a), q_n(t_a))\mathcal{O}_b(p_n(t_b), q_n(t_b))\cdots\end{aligned}}$$

In the second step, go to Green functions. Here use the field notation, since this is what we need

$$\begin{aligned}&\langle \Omega | T(\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\cdots) | \Omega\rangle \\ &= \lim_{\substack{t_i \to -\infty \\ t_f \to \infty}} \int \prod_{\vec{x}} \mathrm{d}\phi_i(\vec{x}) \prod_{\vec{x}} \mathrm{d}\phi_f(\vec{x}) \langle \Omega | \phi_f; t_f\rangle \langle \phi_f; t_f | T(\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\cdots) | \phi_i; t_i\rangle \langle \phi_i; t_i | \Omega\rangle \\ &= \int \mathcal{D}[\phi(x)]\mathcal{D}[\Pi(x)] \langle \Omega | \phi(t_f = \infty, \vec{x})\rangle \langle \phi(t_i = -\infty; \vec{x}) | \Omega\rangle \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x (\Pi(x)\dot{\phi}(x) - \mathcal{H}(\phi(x).\Pi(x)))} \mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\cdots\end{aligned}$$

where $\mathcal{O}_a(x_a)$ and $\mathcal{O}_b(x_b)$ are field operators, $\phi(x_a)$ or products at the same point $\phi^n(x_a)$. At the first equality, we insert a complete set of coordinate eigenstates at t_i, t_f. The right hand side of the first equality corresponds to an integral over all initial and final configurations, so that $\mathcal{D}[\phi(x)]$ is now unconstrained, similar to $\mathcal{D}[\Pi(x)]$. To obtain the final result, we compute $\langle \phi; \mp\infty | \Omega\rangle$. Since $\phi(\vec{x})$ are the “coordinates” of the system, this is the wave function of the vacuum state in the coordinate representation. (Compare $\psi(\vec{x}) = \langle \vec{x} | \psi\rangle$ in quantum mechanics.)

We assume that for $t \to \mp\infty$, the interacting field behaves as a free field. Then

$$\phi(x) \xrightarrow{t\to\mp\infty} \int \frac{\mathrm{d}^3p}{(2\pi)^3 2p^0} \left(\mathrm{e}^{-\mathrm{i}px} a_{\substack{\text{in}\\\text{out}}}(p) + \mathrm{e}^{\mathrm{i}px} a^\dagger_{\substack{\text{in}\\\text{out}}}(p)\right)$$

where $a_{\substack{\text{in}\\\text{out}}}(p)$ and $a^\dagger_{\substack{\text{in}\\\text{out}}}(p)$ are with standard free field commutation relations. And for $t \to \mp\infty$,

$$\Pi(x) = \dot{\phi}(x)$$

This assumption will be motivated better later in the discussion of scattering theory. The above allows us to write

$$a_{\substack{\text{in}\\\text{out}}}(p) = \lim_{t\to\mp\infty} \int \mathrm{d}^3\vec{x}\,\mathrm{e}^{\mathrm{i}px} 2p^0 \left(\phi(x) + \frac{\mathrm{i}}{p^0}\Pi(x)\right)$$

Since $a_{\substack{\text{in}\\\text{out}}}(p)|\Omega = 0$,

$$\begin{aligned} 0 &= \langle\phi;\mp\infty|a_{\substack{\text{in}\\\text{out}}}(p)|\Omega\rangle \\ &= \lim_{t\to\mp\infty} \int \mathrm{d}^3\vec{x}\,\mathrm{e}^{\mathrm{i}px} 2p^0 \langle\phi;\mp\infty|\phi(x) + \frac{\mathrm{i}}{p^0}\Pi(x)|\Omega\rangle \\ &= \lim_{t\to\mp\infty} \int \mathrm{d}^3\vec{x}\,\mathrm{e}^{\mathrm{i}px} 2p^0 \left(\phi(\vec{x}) + \frac{1}{p^0}\frac{\delta}{\delta\phi(\vec{x})}\right) \langle\phi;\mp\infty|\Omega\rangle \end{aligned}$$

Recall $\langle q;t|P_n(t) = -\mathrm{i}\dfrac{\partial}{\partial q_n}\langle q;t|$. Now functional derivative

$$\begin{aligned} P_n &\to \Pi(\vec{x}) \\ \frac{\partial}{\partial q_n} &\to \frac{\delta}{\delta\phi(\vec{x})} \end{aligned}$$

Also

$$\begin{gathered} \psi(q_n,t) = \langle q;t|\psi\rangle \text{ wave function} \\ \downarrow \\ \psi[\phi(\vec{x},t)] = \langle\phi;t|\psi\rangle \text{ wave functional} \end{gathered}$$

So

$$\langle\phi;t|\Pi(t,\vec{x})|\psi\rangle = -\mathrm{i}\frac{\delta}{\delta\phi(\vec{x})}\langle\phi;t|\psi\rangle = -\mathrm{i}\frac{\delta}{\delta\phi(\vec{x})}\psi[\phi(\vec{x},t)]$$

Hence $\langle\phi;\mp\infty|\Omega\langle$ satisfies the first-order functional differential equation

$$\int \mathrm{d}^3\vec{x}\,\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}} \left(\frac{\delta}{\delta\phi(\vec{x})} + p^0\phi(\vec{x})\right)\langle\phi;\mp\infty|\Omega\rangle = 0$$

Compare

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} + ax\right) f(x) = 0$$

which is solved by const $\times\, \mathrm{e}^{-\frac{1}{2}ax^2}$. Hence try the solution

$$\boxed{\langle\phi;\mp\infty|\Omega\rangle = N\exp\left(-\frac{1}{2}\int \mathrm{d}^3\vec{x}\mathrm{d}^3\vec{y}\,K(\vec{x},\vec{y})\phi(\vec{x})\phi(\vec{y})\right)}$$

$\phi(\vec{x})$, $\phi(\vec{y})$ at $t = \mp\infty$. Insert into the functional differential equation

$$\begin{gathered} \int \mathrm{d}^3\vec{x}\,\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}} \left(-\int \mathrm{d}^3\vec{y}\,K(\vec{x},\vec{y})\phi(\vec{y}) + p^0\phi(\vec{x})\right) N\exp(\cdots)\times\langle\phi;\mp\infty|\Omega\rangle \stackrel{!}{=} 0 \\ \int \mathrm{d}^3\vec{x}\,\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}}K(\vec{x},\vec{y}) = p^0\mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{y}} \\ \boxed{K(\vec{x},\vec{y}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}\mathrm{e}^{\mathrm{i}\vec{p}\cdot(\vec{x}-\vec{y})}p^0} \end{gathered}$$

Then

$$\langle\Omega|\phi;\infty\rangle\langle\phi;-\infty|\Omega\rangle = |N|^2\mathrm{e}^{-\frac{1}{2}\int \mathrm{d}^3\vec{x}\mathrm{d}^3\vec{y}\,K(\vec{x},\vec{y})(\phi(\infty,\vec{x})\phi(\infty,\vec{y}) + \phi(-\infty,\vec{x})\phi(-\infty,\vec{y}))}$$

Now use in the exponent

$$f(\infty) + f(-\infty) = \lim_{\varepsilon\to0^+}\varepsilon\int_{-\infty}^{\infty}\mathrm{d}t\,f(t)\mathrm{e}^{-\varepsilon(t)}$$

where $f(t) = \phi(t,\vec{x})\phi(t,\vec{y})$ and ε is infinitesimal and positive. We have used $\varepsilon \mathrm{e}^{-\varepsilon(t)} = \mp \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{-\varepsilon(t)}$ and integration by parts. To get

$$|N|^2 \mathrm{e}^{-\frac{1}{2}\varepsilon \int \mathrm{d}^3\vec{x}\mathrm{d}^3\vec{y} \int_{-\infty}^{+\infty} \mathrm{d}t\, \mathrm{e}^{-\varepsilon|t|}\, K(\vec{x},\vec{y})\phi(t,\vec{x})\phi(t,\vec{y})} = |N|^2 \mathrm{e}^{-\frac{1}{2}\varepsilon \int \mathrm{d}^4x\, \phi(x)^2}$$

This follows from

- inserting the solution for $K(\vec{x},\vec{y})$
- $\varepsilon p^0 = \varepsilon$ since $p^0 > 0$ and ε infinitesimal. Then K simplifies to $K(\vec{x},\vec{y}) = \delta^{(3)}(\vec{x}-\vec{y})$
- $\varepsilon \mathrm{e}^{-\varepsilon(t)} \to \varepsilon$. Since the t-integral extends to $\pm\infty$, it is not obvious at this point that this replacement is correct. We will justify this later when we shall see more explicitly the meaning of the "iε-prescription".

This gives the Hamilton path integral formula for Green functions

$$\boxed{\langle\Omega|T(\mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\cdots)|\Omega\rangle = |N|^2 \int \mathcal{D}[\phi(x)]\mathcal{D}[\Pi(x)]\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\Pi\dot{\phi} - \mathcal{H} + \frac{\mathrm{i}\varepsilon}{2}\phi^2\right)} \mathcal{O}_a(x_a)\mathcal{O}_b(x_b)\cdots}$$

- $|N|^2$ can be determined from $\langle\Omega|\Omega\rangle = 1$.

$$\boxed{|N|^2 = \frac{1}{\int \mathcal{D}[\phi(x)]\mathcal{D}[\Pi(x)]\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \left(\Pi\dot{\phi} - \mathcal{H} + \frac{\mathrm{i}\varepsilon}{2}\phi^2\right)}}}$$

- $\mathcal{H}(x)$ usually contains the term $+\frac{1}{2}m^2\phi(x)^2$. In the following, we do not write the $\frac{\mathrm{i}\varepsilon}{2}\phi^2$ explicitly, but instead we assume that m^2 should be interpreted as $m^2 \to m^2 - \mathrm{i}\varepsilon$.

2.1.3 Lagrange Version of the Path Integral

The exponent $\Pi\dot{\phi} - \mathcal{H}$ suggests the Lagrangian but Π here is an integration variable and not given by $\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}$. However, under certain circumstances, the integral over $\mathcal{D}[\Pi]$ can be done and the path integral assumes a simpler form: if (a) $\mathcal{H}$ is at most quadratic in the $\Pi(x)$ and (b) $\mathcal{O}(\Pi(x),\phi(x))$ do not depend on $\Pi(x)$.

The Gaussian Path Integral

Finite-dimensional Gaussian integral. Let $Q(z) = \frac{1}{2}A_{kl}z_kz_l + B_kz_l + C$ be a quadratic form, $k,l = 1,\ldots,N$. Then

$$\begin{aligned}\int_{-\infty}^{+\infty} \mathrm{d}z_1\cdots\mathrm{d}z_N\, \mathrm{e}^{-Q(z)} &= \left(\det\frac{A}{2\pi}\right)^{-\frac{1}{2}} \mathrm{e}^{\frac{1}{2}[A^{-1}]_{mn}B_mB_n - C} \\ &= \underbrace{\left(\det\frac{A}{2\pi}\right)^{-\frac{1}{2}}}_{=\prod_{k=1}^{N}\sqrt{\frac{2\pi}{\lambda_k}}} \mathrm{e}^{-Q(\bar{z})}\end{aligned}$$

where $\bar{z} = -A^{-1}B$ is the stationary point $\left.\frac{\partial Q}{\partial z_k}\right|_{z=\bar{z}} = 0$ of the quadratic form. λ_k are the eigenvalues of the symmetric matrix A (a degenerate eigenvalue is wanted several times).

We must assume $\det A \neq 0$ so that the inverse exists.

This generalizes to the Gaussian path integral, when the index k becomes continuous, i.e. $z_k \to z(t)$ and $\int \prod_{k=1}^{N} \mathrm{d}z_k \to \int \mathcal{D}[z(t)]$ as well as $A_{kl} \to A(t,t')$, $B_k \to B(t)$:

$$\boxed{\begin{aligned}&\int \mathcal{D}[z(t)] \mathrm{e}^{-\frac{1}{2}\int \mathrm{d}t\mathrm{d}t'\, A(t,t')z(t)z(t') - \int \mathrm{d}t\, B(t)z(t) - C} \\ &= \left(\det \frac{A}{2\pi}\right)^{-\frac{1}{2}} \mathrm{e}^{\frac{1}{2}\int \mathrm{d}t\mathrm{d}t'\, A^{-1}(t,t')B(t)B(t') - C}\end{aligned}}$$

where we can also write $A^{-1}(t,t')B(t)B(t')$ as $A(t,t')\bar{z}(t)\bar{z}(t')$ with

$$\bar{z}(t) = -\int \mathrm{d}t'\, A^{-1}(t,t')B(t')$$

- What is $A(t,t')^{-1}$? $\det A(t,t')$?

Eigenvalue and Determinant

40

Chapter 3

From Green Functions to Scattering Cross Sections

Chapter 4

Renormalization

Chapter 5

Symmetries and Relativistic Particles & Quantum Fields with Spin

Chapter 6

Spinor Fields and Particles with Spin 1/2

Chapter 7

Vector Fields and Gauge Symmetry

Outlook on Contents of the "Advanced Quantum Field Theory"

- Quantization of general gauge theories (so called non-abelian)

  Example of the strong interaction; renormalization and asymptotic freedom
- General relativity as an effective quantum field theory of gravity. Spin-2 particles, Poincaré transformations as gauge transformations
- Quantum theory of spontaneous symmetry breaking. Applications to the electroweak interaction and the Higgs boson and to pion scattering; non-linear representations of symmetry
- Infrared divergences; factorization in quantum chromodynamics; parton distributions and parton scattering
- Anomalies
- Supersymmetry and the minimal supersymmetric Standard Model

This part focues on methods and concepts with relevant examples. There is also a "Theoretical Particle Physics" course which focuses more on phenomena.

Chapter 8

Non-abelian Gauge Symmetry and the Quantization of Gauge Theories

Chapter 9

Spin-2 Fields and General Relativity as an Effective Quantum Field Theory of Gravitation

Chapter 10

Spontaneous Symmetry Breaking

Chapter 11

Electroweak Symmetry Breaking and the Higgs Boson

Chapter 12

Infrared Divergences and the Strong Interaction in High-energy Processes

Chapter 13

Anomalies

Chapter 14

Supersymmetry

Appendix A

Basics on Lie Groups

A.1 Basics on Lie Groups

Definition A.1 (Definition of Group)**.** A set G equipped with an operation $\cdot$ that combines any two elements a, b to form another element $a \cdot b$. $(G, \cdot)$ must satisfy:

(1) closure: $\forall a, b \in G$, $a \cdot b \in G$.

(2) associativity: $\forall a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(3) identity: there exists an element $e \in G$: $\forall a \in G$, $e \cdot a = a \cdot e = a$.

(4) inverse: $\forall a \in G$, there exists $b \in G$ such that $a \cdot b = b \cdot a = e$ (b is denoted a^{-1}).

Definition A.2 (Lie Group)**.** A **Lie group** is a group with ∞ number of elements that is also a differentiable manifold. Any group element continuously connected with $\mathbb{1}$ can be written as

$$U = \mathrm{e}^{\mathrm{i}\alpha^a T^a}\mathbb{1}$$

where α^a are numbers and T^a are the group generators.

If we know the explicit form of the group elements U, we can deduce the form of the T^a by expanding around $\mathbb{1}$.

For example,

Example A.1 (orthochronous Lorentz, $SO(1,3)$)**.** Boost along X^1 axis is

$$\Lambda^\mu{}_\nu = \left(\begin{array}{cc|cc} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

where $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$. We know Lorentz transformation is

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu$$

or we could write it as

$$\begin{cases} t' = \gamma(t + \beta x) \\ x' = \gamma(\beta t + \gamma x) \\ y' = y \\ z' = z \end{cases}$$

expand for small β at $\mathcal{O}(\beta)$,

$$\Lambda^\mu{}_\nu = \left(\begin{array}{cc|cc} 1 & \beta & & \\ \beta & 1 & & \\ \hline & & 1 & 0 \\ & & 0 & 1 \end{array}\right) = \delta^\mu{}_\nu + \beta \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & \\ \hline & & & \\ & & & \end{array}\right)$$

now

$$U \approx 1 + \mathrm{i}\alpha^a T^a$$

so $\alpha^a \to \beta$ and

$$\mathrm{i}T^a \to \omega^\mu{}_\nu = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \\ \hline & & \\ & & \end{array}\right)$$

is the generator of boosts along X^1.

Definition A.3. The generators T^a form a **Lie algebra**, defined through the commutation relations

$$[T^a, T^b] = \mathrm{i}f^{abc}T^c$$

where f^{abc} is called the structure constants.

And the group is

- abelian if $f^{abc} = 0$
- non-abelian, otherwise, e.g. $\mathfrak{su}(2)$ has $f^{abc} = \varepsilon^{abc}$, which is totally anti-symmetric with $\varepsilon^{123} = 1$.

Note that $[A, B] = AB - BA$. Therefore,

$$[A, [B, C]] = [A, BC - CB] = ABC - ACB - BCA + CBA$$

And so we have **Jacobi identity**

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0 \tag{A.1}$$

Now in terms of structure constants, $\begin{cases} A \to a \\ B \to b \\ C \to c \end{cases}$

$$\begin{aligned} [T^a, [T^b, T^c]] &= [T^a, \mathrm{i}f^{bcd}T^d] \\ &= \mathrm{i}f^{bcd}[T^a, T^d] \\ &= \mathrm{i}f^{bcd}\mathrm{i}f^{ade}T^e \\ &= -f^{bcd}f^{ade}T^e \end{aligned}$$

Then (A.1) is

$$(-f^{bcd}f^{ade} - f^{cad}f^{bde} - f^{abd}f^{cde})T^e = 0$$

We get the **Jacobi identity for structure constants**

$$\boxed{f^{bcd}f^{ade} + f^{cad}f^{bde} + f^{abd}f^{cde} = 0}$$

The commutative relations completely determine structure of the group sufficiently close to $\mathbb{1}$; if we go far away then global aspects matter (e.g. $SU(2)$ and $O(3)$, which have same commutative relations but differ globally).

Remark. But this is NOT relevant for introductory description of non-abelian gauge theories.

Note also that f^{abc} are anti-symmetric in first 2 indices:

$$\begin{aligned} [T^a, T^b] &= \mathrm{i}f^{abc}T^c \\ &\parallel \\ -[T^b, T^a] &= -\mathrm{i}f^{bac}T^c \Rightarrow f^{abc} = -f^{bac}. \end{aligned}$$

Definition A.4. An **ideal** is a sub-algebra $I \subset \mathfrak{g}$ such that it is an invariant subalgebra, i.e.

$$[g, i] \in I \text{ for any } \begin{cases} g \in \mathfrak{g} \\ i \in I \end{cases}$$

Definition A.5. $\{0\}$ and the whole $\mathfrak{g}$ are trivial ideals; An algebra that does not admit a non-trivial ideal is a **simple algebra**, e.g. $\mathfrak{su}(N)$, $\mathfrak{so}(N)$.

Definition A.6. A **semi-simple algebra** is the direct sum of simple algebras, e.g. Standard model $\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$.

Theorem A.1. All finite-dimensional representation of semi-simple algebras are Hermitian.

Remark. It means that we can construct theories where the symmetry is a unitary transform as fields. [unitary theories couple probabilities]

We are also interested in the case where the number of generators is FINITE, which means compact algebras. (Because the corresponding Lie group is a finite-dimensional compact manifold).
The requirement of being SIMPLE and COMPACT is very stringent: **Classification**.

- unitary groups $U(N)$: defining representation acts on space of N-dimensional complex vectors $U = \mathrm{e}^{\mathrm{i}\alpha T}$ with T Hermitian, $T^\dagger = T$. Then

$$\begin{aligned} U^\dagger U &= (\mathrm{e}^{\mathrm{i}\alpha T})^\dagger (\mathrm{e}^{\mathrm{i}\alpha T}) = \mathrm{e}^{-\mathrm{i}\alpha T^\dagger} \mathrm{e}^{\mathrm{i}\alpha T} \\ &= \mathrm{e}^{-\mathrm{i}\alpha T + \mathrm{i}\alpha T} = \mathbb{1} \end{aligned}$$

then a complex inner product is preserved: take ψ, χ states, then

$$[U\psi]^\dagger U\chi = \psi^\dagger \underbrace{U^\dagger U}_{\mathbb{1}} \chi = \psi^\dagger \chi$$

now $U(N)$ contains the **pure phase** transformations $U = \mathrm{e}^{\mathrm{i}\alpha}$, which form a $U(1)$ subgroup. These are removed by requiring $\det U = 1$ (and not a complex phase with $|\cdot| = 1$) which gives the **simple group** $SU(N)$ whose dimension is

$$d(SU(N)) = N^2 - 1$$

because count real parameters:

$N \times N$ complex matrix	$2N^2$
$U^\dagger U = \mathbb{1}$	N^2 conditions
$\det U = 1$	1 condition

$\Rightarrow$
$$2N^2 - N^2 - 1 = N^2 - 1$$

- orthogonal groups: preserve a real inner product. Let ψ, χ be vectors, then

$$O\psi \cdot O\chi = \psi^{\mathrm{T}} \underbrace{O^{\mathrm{T}} O}_{\mathbb{1}} \chi = \psi^{\mathrm{T}} \chi$$

How many generators? Real $N \times N$ matrix satisfying $O^T O = \mathbb{1}$ symmetric, so $N + \dfrac{N^2 - N}{2}$ independent conditions.

$$\begin{aligned} d(O(N)) &= N^2 - \left[N + \frac{N^2 - N}{2}\right] \\ &= \frac{N(N-1)}{2} \end{aligned}$$

Now $\det 0 = \pm 1 \to$ take $+1$ to get $SO(N)$ rotation group in N dimensions. Then it is clear that

$$d(SO(N)) = \frac{N(N-1)}{2}$$

is the number of planes in N dimensions.

- symplectic groups $\mathrm{Sp}(N)$, where N even: preserve an anti-symmetric inner product

$$\psi^{\mathrm{T}} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \chi$$

where we denote $J \equiv \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$ in $\frac{N}{2} \times \frac{N}{2}$ block form. It satisfies

$$S^{\mathrm{T}} J S = J$$

Then $\frac{N^2 - N}{2}$ is the number of conditions from

$$S^{\mathrm{T}} J S = J$$

$S^{\mathrm{T}} J S$ is anti-symmetric since

$$(S^{\mathrm{T}} J S)^{\mathrm{T}} = S^{\mathrm{T}} J^{\mathrm{T}} S = -S^{\mathrm{T}} J S$$

Therefore the dimension of symplectic group is

$$\begin{aligned} d(\mathrm{Sp}(N)) &= N^2 - \left[\frac{N^2 - N}{2}\right] \\ &= \frac{N(N+1)}{2} \end{aligned}$$

- exceptional groups (5 of them): G_2, F_4, F_6, F_7, F_8.

  The subscript means the rank of the group=dimension of Cartan subgroup, which is the max commuting subgroup; Equivalently, number of degenerate generators. [$SU(N)$ has rank $N-1$.]

Remark. This completes classification of compact simple Lie algebras.

Note also that if there is any Hermitian representation (as for semi-simple algebras), then the structure constants are real:

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c \quad \Rightarrow \quad \begin{array}{c} [T^a, T^b]^\dagger = -\mathrm{i} f^{abc*} T^c \\ \| \\ (T^a T^b - T^b T^a)^\dagger \\ \| \\ [T^b, T^a] = \mathrm{i} f^{bac} T^c = -\mathrm{i} f^{abc} T^c \end{array}$$

f^{abc} is anti-symmetric in first 2 indices, so

$$f^{abc} = f^{abc*}$$

A.2 General Properties of Representations

Representations of algebra are constructed by embedding generators into matrices. One can show that for compact semi-simple Lie algebra, for any representation R, we can choose a basis for the generators such that

$$\mathrm{Tr}(T_R^a T_R^b) = T(R) \delta^{ab}$$

where $T(R)$ constant for each R. Now combined with

$$[T_R^a, T_R^b] = \mathrm{i} f^{abc} T_R^c$$

this implies

$$[T_R^a, T_R^b] T_R^c = \mathrm{i} f^{abd} T_R^d T_R^c$$

Take trace

$$\begin{aligned} \mathrm{Tr}[[T_R^a, T_R^b], T_R^c] &= \mathrm{i} f^{abd}\, \mathrm{Tr}[T_R^d T_R^c] \\ &= \mathrm{i} f^{abd} T(R) \delta^{dc} = \mathrm{i} T(R) f^{abc} \end{aligned}$$

$$\Rightarrow \qquad f^{abc} = -\frac{\mathrm{i}}{T(R)}\, \mathrm{Tr}[[T_R^a, T_R^b], T_R^c]$$

Now this implies that

$$f^{abc} = f^{bca}$$

because

$$\begin{aligned}\mathrm{Tr}[[T^a_R, T^b_R], T^c_R] &= \mathrm{Tr}[T^a_R T^b_R T^c_R - T^b_R T^a_R T^c_R] \\ &= \mathrm{Tr}[T^b_R T^c_R T^a_R - T^c_R T^b_R T^a_R] \\ &= \mathrm{Tr}[[T^b_R, T^c_R] T^a_R]\end{aligned}$$

Now recall that $f^{abc} = -f^{bac} \Rightarrow$ combined, they tell us that f^{abc} are completely anti-symmetric.

Now for a given R, infinitesimal transform under the group is

$$\begin{aligned}\phi &\to (1 + \mathrm{i}\alpha^a T^a_R)\phi \quad \text{take complex conjugate} \\ \phi^* &\to (1 - \mathrm{i}\alpha^a T^{a*}_R)\phi^*\end{aligned}$$

But also we can define the conjugate representation

$$\phi^* \to (1 + \mathrm{i}\alpha^a T^a_{\overline{R}})\phi^*$$

hence

$$T^a_{\overline{R}} = -(T^a_R)^* = -(T^a_R)^{\mathrm{T}}$$

Now $\overline{R}$ is equivalent to R, if there exists a unitary transform U such that

$$T^a_{\overline{R}} = U T^a_R U^\dagger$$

In this case, R is a real representation. Then we can always find a matrix E_{ab} such that if $\psi, \chi \in R$ then $\psi_a E_{ab} \chi_b$ is invariant. If

$$E_{ab} \begin{cases} \text{symmetric} & \text{strictly real} \\ \text{anti-symmetric} & \text{pseudo-real} \end{cases}$$

A.3 Representation of $SU(N)$ Groups

Free theory of N complex massless fields is automatically invariant under $U(N) \cong SU(N) \times U(1)$. Hence the importance of $SU(N)$ groups. The two most important representations are fundamental and adjoint.

- Fundamental (or defining): acts on space of N-dimensional complex vectors smallest non-trivial representation of the algebra. Dimension is N. For $SU(N)$, set of $N \times N$ Hermitian, traceless matrices.

  [Why traceless? Consider $U(\vec{\alpha}) = \mathrm{e}^{\mathrm{i}\vec{\alpha}\cdot\vec{T}}$ and diagonalize (Hermitian matrices are diagonalizable)

  $$V\vec{\alpha} \cdot \vec{T} V^{-1} = D$$

  where D is diagonal, then

  $$\begin{aligned}\det U &\xlongequal{\text{det is basis-independent}} \det \mathrm{e}^{\mathrm{i}D} \\ &= \prod_j \mathrm{e}^{\mathrm{i}[D]_{jj}} \\ &= \mathrm{e}^{\mathrm{i}\sum_j [D]_{jj}} \\ &= \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}\, D} \\ &\xlongequal{\text{Tr is basis-independent}} \mathrm{e}^{\mathrm{i}\,\mathrm{Tr}(\vec{\alpha}\cdot\vec{T})}\end{aligned}$$

  hence if $\mathrm{Tr}\, T^a = 0 \Rightarrow \det U = 1$. Note that tracelessness argument applies for any semisimple algebra.]

  - For $SU(2)$, $T^a = \dfrac{\sigma^a}{2}$. σ^a are Pauli matrices

  $$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

they satisfy

$$\begin{aligned}[T^a, T^b] &= \frac{1}{4}[\sigma^a, \sigma^b] \\ &= \frac{1}{4} 2\mathrm{i}\varepsilon^{abc}\sigma^c \\ &= \mathrm{i}\varepsilon^{abc}T^c\end{aligned}$$

where ε^{abc} are structure constants.

– For $SU(3)$, $T^a = \dfrac{l^a}{2}$, where l^a are Gell-Mann matrices. Now for the 2 of $SU(2)$, we have

$$T^a_R = \frac{\sigma^a}{2} \qquad T^a_{\overline{R}} = -\frac{\sigma^{a*}}{2}$$

But we know that

$$\sigma^2\sigma^a\sigma^2 = -\sigma^{a*}$$

hence

$$T^a_{\overline{R}} = -\frac{\sigma^{a*}}{2} = \sigma^2\left(\frac{\sigma^a}{2}\right)\sigma^2 = \sigma^2 T^a_R \sigma^2$$

i.e. real representation with $U = \sigma^2$. Also, $E_{ab} = (\mathrm{i}\sigma^2)_{ab} = \varepsilon_{ab}$, so it is pseudo-real.
For $N > 2$, the N of $SU(N)$ is instead complex (so in particularly, $3 \neq \overline{3}$ for $SU(3)$).

- Adjoint: acts on the space spanned by the generators themselves

$$(T^a_{\text{adj}})_{bc} = qf^{abc}$$

where q purely imaginary. The dimension is $N^2 - 1$ (number of generators), e.g. for $SU(2)$:

$$(T^a_{\text{adj}})_{bc} = q\varepsilon^{abc} \to T^1_{\text{adj}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \text{ etc.}$$

As an exercise, find q such that T^a_{adj} satisfy commutative relations of algebra.

Remark. Adjoint representation if important because it is the representation where non-abelian gauge fields transform.

Note that the conjugate representation of the adjoint is

$$T^a_{\overline{\text{adj}}} = -(T^a_{\text{adj}})^* = -(qf^{abc})^* = +qf^{abc} = T^a_{\text{adj}}$$

hence the adjoint is always real

$$U = \mathbb{1}.$$

A.4 Casimir Operator

For any representation T^a_R, we know that $T^2 \equiv T^a_R T^a_R$ (sum over a), commutes with all the generators:

$$\begin{aligned}[T^2, T^b_R] &= [T^a_R T^a_R, T^b_R] \\ &= T^a_R[T^a_R, T^b_R] + [T^a_R, T^b_R]T^a_R \\ &= T^a_R \mathrm{i} f^{abc} T^c_R + \mathrm{i} f^{abc} T^c_R T^a_R \\ &= \mathrm{i} \underbrace{f^{abc}}_{\text{anti-symmetric}} \underbrace{\{T^a_R, T^c_R\}}_{\text{symmetric}} = 0\end{aligned}$$

where we have used $[AB, C] = A[B, C] + [A, C]B$ at the second equality. Hence T^2 must be propositional to the identity, by Schur's lemma. So

$$T^a_R T^a_R = c_2(R)\mathbb{1} \tag{A.2}$$

where $c_2(R)$ is quadratic Casimir of R.

[Familiar example: $SU(2)$, $J^2 T^a_R T^a_R$ is Casimir with eigenvalues $j(j+1)$, which is total spin.]

As we discussed, we also have for an appropriate choice of basis

$$\mathrm{Tr}(T_R^a T_R^b) = T(R)\delta^{ab}$$

where $T(R)$ is the index of R (sometimes called $c(R)$). Now contract this with δ^{ab}. From (A.2),

$$\begin{aligned}&\mathrm{Tr}(T_R^a T_R^a) = T(R)d(G)\\ &\quad\ \ \|\\ &c_2(R)d(R)\end{aligned}$$

$$\Rightarrow \qquad d(R)c_2(R) = T(R)d(G)$$

which is useful to compute $c_2(R)$.

- Fundamental of $SU(N)$, typical to choose $T(\text{fund}) = \dfrac{1}{2} \equiv T_{\mathrm{F}}$ as in the case of $SU(2)$, then

  $$c_2(\text{fund}) = \frac{T(\text{fund})d(G)}{d(\text{fund})} = \frac{1}{2}\frac{N^2-1}{N} \equiv c_{\mathrm{F}}$$

  which c_{F} is often used.

- Adjoint of $SU(N)$: first note that $d(R) = d(G)$, so $c_2(\text{adj}) = T(\text{adj})$. How do we compute $c_2(\text{adj})$? Instructive to do it by building adjoint as product of fund and antifund: given 2 representation R_1 and R_2, their **direct product** is a representation of dimension $d(R_1) \cdot d(R_2)$. Objects transforming under it are **tensors** ξ_{pq}, where $p \in R_1$ and $q \in R_2$. The product representation can in general be decomposed as direct sum of irrepresentation:

  $$R_1 \times R_2 = \sum_i R_i \tag{A.3}$$

  The matrices are

  $$t^a_{R_1\times R_2} = t^a_{R_1} \otimes \mathbb{1} + \mathbb{1} \otimes t^a_{R_2}$$

  Now the quadratic Casimir of the product is

  $$t^a_{R_1\times R_2} t^a_{R_1\times R_2} = (t^a_{R_1})^2 \otimes \mathbb{1} + \mathbb{1} \otimes (t^a_{R_2})^2 + 2t^a_{R_1} \otimes t^a_{R_2}$$

  take the trace

  $$\begin{aligned}\mathrm{Tr}({t^a_{R_1\times R_2}}^2) &= \mathrm{Tr}({t^a_{R_1}}^2)d(R_2) + d(R_1)\,\mathrm{Tr}({t^a_{R_2}}^2) + 0\\ &= c_2(R_1)d(R_1)d(R_2) + d(R_1)c_2(R_2)d(R_2)\\ &= [c_2(R_1) + c_2(R_2)]d(R_1)d(R_2)\end{aligned} \tag{A.4}$$

  But also, from (A.3) we get

  $$\mathrm{Tr}({T^a_{R_1\times R_2}}^2) \xlongequal{\text{block-diag form}} \sum_i \mathrm{Tr}(t^2_{R_i}) = \sum_i c_2(R_i)d(R_1) \tag{A.5}$$

  Now apply to case of $R_1 = N$, $R_2 = \overline{N}$ in $SU(N)$:

  $$N \times \overline{N} = 1 + (N^2 - 1)$$

  For 1 is $\xi_{pq} = \delta_{pq}$ and (N^2-1) is remaining pieces: $N \times N$ traceless tensor, which is adjoint representation. Then from (A.4)=(A.5) get

  $$[c_2(N) + c_2(\overline{N})]N^2 = c_2(\text{adj})d(\text{adj})$$

  for the singlet representation, we have $T_1^a = 0$, so $c_2(1) = 0$. Then

  $$2c_2(N)N^2 = c_2(\text{adj})(N^2-1)$$

  By apply the above $c_2(N)$, we have

  $$\Rightarrow \qquad c_2(\text{adj}) = 2N^2\frac{N^2-1}{2N}\frac{1}{N^2-1} = N \equiv c_A$$

Hence

$$c_2(\text{adj}) = c_A = N$$
$$T(\text{adj}) = T_A = N$$

So we have seen for the fundamental of $SU(N)$ that

$$\begin{aligned} \mathrm{Tr}(t^a t^b) &= T_\mathrm{F}\delta^{ab} \qquad T_\mathrm{F} = \frac{1}{2} \\ \textstyle\sum_a (t^a t^a)_{ij} &= c_\mathrm{F}\delta_{ij} \quad c_\mathrm{F} = \frac{N^2-1}{2N} \end{aligned}$$

Another useful relation is

$$\boxed{\sum_a (t^a)_{ij}(t^a)_{ue} = T_\mathrm{F}\left(\delta_{ie}\delta_{ju} - \frac{1}{N}\delta_{ij}\delta_{ue}\right)}$$

called the "completeness relation". To prove it, begin with the fact that for any $N \times N$ complex matrix M, we can write

$$M = M_0\mathbb{1} + M^a t^a \tag{A.6}$$

where M_0 and M^a are complex coefficients. t^a is traceless. Then

$$\begin{aligned} \mathrm{Tr}(M) &= M_0 N \\ \mathrm{Tr}(Mt^b) &= \mathrm{Tr}((M_0\mathbb{1} + M_a^a t^a)t^b) = M_a \underbrace{\mathrm{Tr}(t^a t^b)}_{=T_\mathrm{F}\delta^{ab}} = T_\mathrm{F} M^b \end{aligned}$$

$$\Rightarrow \qquad \begin{cases} M_0 &= \frac{1}{N}\mathrm{Tr}(M) \\ M^b &= \frac{1}{T_\mathrm{F}}(Mt^b) \end{cases}$$

Now plug trace here (A.6)

$$\begin{aligned} M &= \frac{1}{N}\mathrm{Tr}(M)\mathbb{1} + \frac{1}{T_\mathrm{F}}\mathrm{Tr}(Mt^a)t^a \\ M_{ij} &= \frac{1}{N}M_{kk}\delta_{ij} + \frac{1}{T_\mathrm{F}}M_{eu}(t^a)_{ue}(t^a)_{ij} \end{aligned}$$

This implies

$$\delta_{ie}\delta_{ju}M_{eu} = \frac{1}{N}M_{eu}\delta_{eu}\delta_{ij} + \frac{1}{T_\mathrm{F}}M_{eu}(t^a)_{ue}(t^a)_{ij}$$

now since M is an arbitrary matrix, the coefficient of M_{eu} must vanish.

$$\Rightarrow \qquad \delta_{ie}\delta_{ju} = \frac{1}{N}\delta_{ij}\delta_{ue} + \frac{1}{T_\mathrm{F}}(t^a)_{ij}(t^a)_{ue}$$

which is what we wanted. Another useful result is

$$\boxed{\sum_{a,b} f^{abc}f^{abd} = c_A\delta^{cd}}$$

which is just the determined of the quadratic Casimir for the adjoint:

$$(T^a T^a)_{cd} = c_2(\text{adj})\delta_{cd}$$

but now we saw that $(T^a)_{bc} = qf^{abc}$, $q \in \mathrm{Im}$, then

$$\begin{aligned} (T^a T^a)_{cd} &= (T^a)_{cb}(T^a)_{bd} \\ &= qf^{acb}qf^{abd} \\ &= -f^{acb}f^{abd} \\ &= +f^{abc}f^{abd} \end{aligned}$$

as derived. These results will be useful for calculations in non-abelian gauge theories.

Appendix B

Transversity of Gauge Boson Self-energy from BRST Symmetry

Our starting point is the gauge-fixed Lagrangian, which for the purpose of exploiting BRST is best written using the auxiliary bosonic field B^a: in a covariant gauge, we have

$$\boxed{\widetilde{\mathcal{L}} = \underset{\substack{\uparrow \\ \text{gauge+matter}}}{\mathcal{L}} + B^a \partial_\mu A^{\mu a} + \frac{\xi}{2}(B^a)^2 + (\partial_\mu \bar{c})^a (D^\mu c)^a}$$

where we use the convention $D_\mu = \partial_\mu - \mathrm{i} g A_\mu$ then

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - g f^{abc} A_\mu^c$$

Let us see explicitly how this $\widetilde{\mathcal{L}}$ arises: Start from naive functional integral

$$\langle \Omega | T\{O_i(X_i) O_j(X_j) \cdots\} | \Omega \rangle = |N|^2 \int DA \mathrm{e}^{\mathrm{i}S[A]} O_i O_j \cdots \tag{B.1}$$

(B.1) throughout the discussion, "T" actually means "T^*", as usual when using the path integral. This allows us in particular to pull derivatives in front of the T. Now we must eliminate the redundancy through a gauge fixing (pick one representative for each gauge orbit). Insert identity as

$$1 = \int D\varepsilon \det\left(\frac{\delta f[A_\varepsilon]}{\delta \varepsilon}\right) \delta(f[A_\varepsilon] - F) \tag{B.2}$$

where f will be our gauge-fixing functions, and A_ε is the gauge field transformed by a small ε:

$$A^a_{\mu\varepsilon} = A^a_\mu + \frac{1}{g} D^{ac}_\mu \varepsilon^c$$

then

$$= |N|^2 \int DA\, D\varepsilon\, \mathrm{e}^{\mathrm{i}S[A]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta \varepsilon}\right) \delta(f[A_\varepsilon] - F) O_i O_j \cdots$$

now S and $O_i O_j \cdots$ are gauge invariant, and also the measure satisfies $DA = DA_\varepsilon$, so

$$= |N|^2 \int DA_\varepsilon\, D_\varepsilon \mathrm{e}^{\mathrm{i}S[A_\varepsilon]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta \varepsilon}\right) \delta(f[A_\varepsilon] - F) O_i O_j \cdots$$

redefining $A_\varepsilon \to A$ as integration variable, we obtain

$$= |N|^2 \left(\int D\varepsilon\right) \int DA\, \mathrm{e}^{\mathrm{i}S[A]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta \varepsilon}\right)_{\varepsilon=0} \delta(f[A] - F) O_i O_j \cdots \tag{B.3}$$

$$= |N'|^2 \int DA\, \mathrm{e}^{\mathrm{i}S[A]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta \varepsilon}\right)_{\varepsilon=0} \delta(f[A] - F) O_i O_j \cdots \tag{B.4}$$

where in the last step we used the fact that the integral over $D\varepsilon$ just gives an (infinite) overall constant.

Now the current expression does not depend on F [recall (B.2)], so let us multiply times the constant $\int DF\,\widetilde{G}[F]$ with a functional $\widetilde{G}[F]$ of our choice. Pick them

$$\widetilde{G}[F] = \int DB^a \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, B^a\left(F^a + \frac{\xi}{2}B^a\right)}$$

which in fact is equivalent to the choice, we made then deriving $\widetilde{\mathcal{L}}$ in the formulation without auxiliary field: complete the square

$$\widetilde{G}[F] = \underbrace{\int DB^a\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x \frac{\xi}{2}\left(B^a + \frac{F^a}{\xi}\right)^2}}_{=\text{constant}}\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\left(-\frac{F^{a\,2}}{2\xi}\right)}$$
$$= \text{constant}\cdot \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\left(-\frac{F^{a\,2}}{2\xi}\right)} = \text{constant}\cdot G[F]$$

where $G[F]$ leads to the formulation without B^a. Then (B.4) becomes

$$|N'|^2\int DA\mathrm{e}^{\mathrm{i}\mathrm{d}^4x\,\mathcal{L}}\det\left(\frac{\delta f[A_\varepsilon]}{\delta\varepsilon}\right)_\varepsilon \delta(f[A]-F)\int DF\,DB^a\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\,B^a\left(F^a+\frac{\xi}{2}B^a\right)}O_iO_j\cdots$$
$$=|N'|\int DA\,DB^a\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\left\{\mathcal{L}+B^aF^a+\frac{\xi}{2}(B^a)^2\right\}}\det\left(\frac{\delta f[A_\varepsilon]}{\delta\varepsilon}\right)_{\varepsilon=0}O_iO_j\cdots$$

Now write the determinant as

$$\det\left(\frac{\delta f[A_\varepsilon]}{\delta\varepsilon}\right)_{\varepsilon=0} = \int D[c^a,\bar{c}^a]\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\mathrm{d}^4y\,\bar{c}^a(x)(-\mathrm{i})\left.\frac{\delta f^a[A_\varepsilon(x)]}{\delta\varepsilon^b(y)}\right|_{\varepsilon=0}c^b(y)}$$

and for the covariant gauges, $f^a = \partial_\mu A^{\mu a}$, obtain

$$\left.\frac{\delta f^a[A_\varepsilon(x)]}{\delta\varepsilon^b(y)}\right|_{\varepsilon=0} = \frac{1}{g}\partial_\mu D^{\mu ab}\delta^{(4)}(x-y)$$

so

$$\mathrm{i}\int \mathrm{d}^4x\,\mathrm{d}^4y\,\bar{c}^a(x)(-\mathrm{i})\frac{\delta f^a[A_\varepsilon(x)]}{\delta\varepsilon^b(y)}c^b(y) = \mathrm{i}\int \mathrm{d}^4x\bar{c}^a(x)(-\mathrm{i})\frac{1}{g}\partial_\mu D^{\mu ab}c^b(x)$$

integration by parts and redefine $\frac{\mathrm{i}c^a}{g}\to c^a$

$$= \mathrm{i}\int \mathrm{d}^4x(\partial_\mu\bar{c}^a)(D^\mu c)^a$$

hence the path integral becomes

$$= |N'|^2\int DA\;DB^a\;D[c,\bar{c}]\;\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\left\{\mathcal{L}+B^a\partial_\mu A^{\mu a}+\frac{\xi}{2}(B^a)^2+\partial_\mu\bar{c}^a(D^\mu c)^a\right\}}O_iO_j\cdots$$

where $\mathcal{L}+B^a\partial_\mu A^{\mu a}+\frac{\xi}{2}(B^a)^2+\partial_\mu\bar{c}^a(D^\mu c)^a = \widetilde{\mathcal{L}}$ is the gauge-fixed Lagrangian. So far we did not include fields. Their effect simply goes into $\mathcal{L}$, provided the integration measure for fermions is invariant under $\psi\to\psi_\varepsilon$. Proceed under this assumption (non-anomalous gauge symmetry).

The gauge-fixed Lagrangian is invariant under BRST transformations $\delta_\theta\widetilde{\mathcal{L}} = 0$, where the action of BRST is $\phi\to\phi+\underbrace{\theta\Delta\phi}_{\delta_\theta\phi}$, for any ϕ (θ =Grassmann number, Δ =Slavnov operator). The action an individual fields is

$$\begin{cases} \delta_\theta \psi = \mathrm{i} g \theta c^a T^a \psi \\ \delta_\theta A^a_\mu = \theta (D_\mu c)^a \\ \delta_\theta c^a = -\dfrac{1}{2} g \theta f^{abc} c^b c^c \\ \delta_\theta \bar{c}^a = \theta B^a \\ \delta_\theta B^a = 0 \end{cases}$$

Δ is nilpotent, $\delta_\theta(\Delta\phi) = 0$. Now invariance under BRST gives rise to Ward identities obeyed by the Green's functions of the non-abelian gauge theory:

$$\sum_{k=1}^{m} \langle \Omega | T\{\phi_{n_1}(x_1) \cdots \phi_{n_{k-1}}(x_{k-1}) \theta \Delta(\phi_{n_k}(x_k)) \phi_{n_{k+1}}(x_{k+1}) \cdots \phi_{n_m}(x_m)\} | \Omega \rangle = 0 \tag{B.5}$$

where ϕ_n are any of the fields in the theory:

$$\psi, A_\mu, c, \bar{c}, B, \ldots$$

This identity is obtained assuming BRST is non-anomalous, which happens if the gauge symmetry is non-anomalous.

We will also exploit the following EOM identities,

$$\begin{aligned} &\left\langle \Omega \middle| T \left\{ \frac{\delta S[\phi_n 1]}{\delta \phi_n(x)} \phi_{n_1} \cdots \phi_{n_m}(x_m) \right\} \middle| \Omega \right\rangle \\ =&\mathrm{i} \sum_{i=1}^{m} \delta^{(4)}(x - x_i) \delta_{n n_i} \langle \Omega | T\{\phi_{n_1}(x_1) \cdots \widehat{\phi}_{n_i}(x_i) \cdots \phi_{n_m}(x_m)\} | \Omega \rangle \end{aligned} \tag{B.6}$$

where $\widehat{\phi}_{n_i}(x_i)$ means "omitted from list" and the summation vanishes unless $n = n_i$ and $x = x_i$ for one of the $\phi_{n_1}, \ldots, \phi_{n_m}$. These identities can be obtained by appropriate manipulations of the generating functional $Z[\phi_n]$. Finally, preparations are finished and we can move on to our main goal. We will apply the Ward identities of BRST (and EOM identities) to prove that the gauge boson self-energy is transverse. Take the two-point function

$$\langle \Omega | T\{A^a_\mu(x) A^b_\nu(y)\} | \Omega \rangle = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathrm{e}^{-\mathrm{i} k \cdot (x-y)} G^{ab}_{\mu\nu}(k)$$

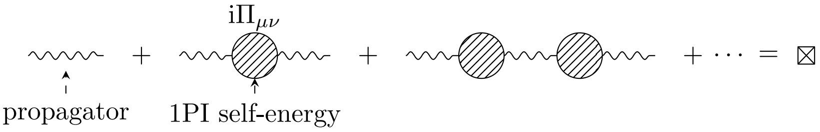
where the most general form of the Fourier transform is

$$G^{ab}_{\mu\nu}(k) = \delta^{ab} \frac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon} \left[\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right) A(k^2) - \frac{k_\mu k_\nu}{k^2} B(k^2) \right]$$

At lowest order in coupling constant g, this is the propagator, so we know

$$\begin{cases} A = 1 + \mathcal{O}(g^2) \\ B = \xi + \mathcal{O}(g^2) \end{cases}$$

The propagator is $-\dfrac{\mathrm{i}}{k^2 + \mathrm{i}\varepsilon} \left(g_{\mu\nu} - (1-\xi) \dfrac{k_\mu k_\nu}{k^2} \right)$. We will now show that $B = \xi$ at all orders in g. If this is true, then we can write the two-point function as



where

$$\Pi_{\mu\nu} = (k^2 g_{\mu\nu} - k_\mu k_\nu) \Pi(k^2) + k_\mu k_\nu \Pi_2(k^2)$$

It's easy to convince oneself that the fact that B does not get corrected at any order in g, is equivalent to

$$\Pi_2 = 0 \tag{B.7}$$

The self-energy is transverse, $k^\mu \Pi_{\mu\nu} = 0$.

Then it is also easy to reuse the series,

$$\begin{aligned}\boxtimes &= \frac{\mathrm{i}}{k^2}\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)(1+\Pi+\Pi^2+\cdots) + \frac{\mathrm{i}}{k^2}\left(-\xi\frac{k_\mu k_\nu}{k^2}\right)\\ &= \frac{\mathrm{i}}{k^2}\left[\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{1-\Pi} - \xi\frac{k_\mu k_\nu}{k^2}\right]\end{aligned}$$

where $-\xi\dfrac{k_\mu k_\nu}{k^2}$ is from the first term only. Hence $A(k^2) = \dfrac{1}{1-\Pi(k^2)}$. Now let us prove that $B = \xi$ at all orders. (B.7) because the first, which already gives $B = \xi$!

Use EOM identity in the form

$$\begin{aligned}0 &= \left\langle \Omega \middle| \left\{ \frac{\delta S}{\delta B^a(x)} \partial^\mu A^b_\mu(y) \right\} \middle| \Omega \right\rangle\\ &= \langle \Omega | T\{(\partial_\nu A^{\nu a}(x) + \xi B^a(x))\partial^\mu A^b_\mu(y)\}|\Omega\rangle\end{aligned}$$

where the first equality comes from that fields do not match, RHS of (B.6) is zero and we insert $\dfrac{\delta S}{\delta B^a(x)} = \partial_\nu A^{\nu a}(x) + \xi B^a(x)$ at the second equality. Which we rewrite as

$$\begin{aligned}\partial^x_\nu \partial^y_\mu \langle \Omega | T\{A^{\nu a}(x) A^{\mu b}(y)\}|\Omega\rangle &= \xi \langle \Omega | T\{B^a(x)\partial^\mu A^b_\mu(y)\}|\Omega\rangle\\ &= -\xi \langle \Omega | T\{\Delta \bar{c}^a(x)\partial^\mu A^b_\mu(y)\}|\Omega\rangle\end{aligned}$$

where $B^a(x) = \Delta \bar{c}^a(x)$. Now look at the above equation: the LHS is already $\partial_\mu\partial_\nu$ (two-point function), but we need to work on the RHS. Let us use the Ward identity for 2 fields: (B.5) reads

$$\begin{aligned}0 &= \langle \Omega | T\{\theta \Delta \bar{c}^a(x)\partial^\mu A^b_\mu(y)\}|\Omega\rangle + \langle \Omega | T\{\bar{c}^a(x)\theta\Delta(\partial^\mu A^b_\mu(y))\}|\Omega\rangle\\ &= \langle \Omega | T\{\theta \Delta \bar{c}^a(x)\partial^\mu A^a_\mu(y)\}|\Omega\rangle - \langle \Omega | T\{\theta\bar{c}^a(x)\Delta(\partial^\mu A^b_\mu(y))\}|\Omega\rangle\end{aligned}$$

now

$$\Delta(\partial^\mu A^b_\mu(y)) = \partial^\mu(\Delta A^b_\mu(y)) = \partial^\mu(D_\mu c(y))^b$$

but also,

$$\frac{\delta S}{\delta \bar{c}^b(y)} = -\partial^\mu D^{bc}_\mu c^c(y)$$

where we integrate by parts before taking derivative. Hence

$$0 = \langle \Omega | T\{\Delta \bar{c}^a(x)\partial^\mu A^b_\mu(y)\}\Big|\Omega\rangle + \left\langle \Omega | T\left\{\bar{c}^a(x)\frac{\delta S}{\delta \bar{c}^b(y)}\right\}|\Omega\right\rangle$$

and $=$ becomes

$$\begin{aligned}\partial^x_\nu \partial^y_\mu \langle \Omega | T\{A^{\nu a}(x) A^{\mu b}(y)\}|\Omega\rangle &= +\xi \left\langle \Omega \middle| T\left\{\bar{c}^a(x)\frac{\delta S}{\delta \bar{c}^b(y)}\right\}\middle|\Omega\right\rangle\\ &= -\xi \left\langle \Omega \middle| T\left\{\frac{\delta S}{\delta \bar{c}^b(y)}\bar{c}^a(x)\right\}\middle|\Omega\right\rangle\end{aligned}$$

$\left\langle \Omega \middle| T\left\{\dfrac{\delta S}{\delta \bar{c}^b(y)}\bar{c}^a(x)\right\}\middle|\Omega\right\rangle$ is precisely the case where the RHS of the EOM identities of (B.6) does not vanish! We arrive at

$$\partial^x_\nu \partial^y_\mu \langle \Omega | T A^{\nu a}(x) A^{\mu b}(y)|\Omega\rangle = -\xi \mathrm{i}\delta^{(4)}(x-y)\delta^{ab}\underbrace{\langle\Omega|\Omega\rangle}_{=1}$$

The rest is easy: write

$$\partial^x_\nu \partial^y_\mu \int \frac{\mathrm{d}^4 k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k\cdot(x-y)} G^{ab}_{\mu\nu}(k) = -\mathrm{i}\xi\delta^{ab}\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}$$

where $\int \dfrac{\mathrm{d}^4 k}{(2\pi)^4}\mathrm{e}^{-\mathrm{i}k\cdot(x-y)}$ is integral representation of $\delta^{(4)}$.

$$\begin{aligned}(-\mathrm{i}k^\nu)(+\mathrm{i}k^\mu)G^{ab}_{\mu\nu}(k) &= -\mathrm{i}\xi\delta^{ab}\\ k^\mu k^\nu G^{ab}_{\mu\nu}(k) &= -\mathrm{i}\xi\delta^{ab}\end{aligned}$$

then recall form of $G^{ab}_{\mu\nu}$:

$$k^\mu k^\nu G^{ab}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2}\delta^{ab}(-k^2 B) = -\mathrm{i}\xi\delta^{ab} \quad \Rightarrow \quad B = \xi$$

which is an exact result.

Some more justification for (B.3):

$$\int DA_\varepsilon \, D\varepsilon \, \mathrm{e}^{\mathrm{i}S[A_\varepsilon]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta\varepsilon}\right) \delta(f[A_\varepsilon] - F) O_i O_j \cdots =$$

where

$$\frac{\delta f^a[A_\varepsilon]}{\delta\varepsilon^b} = \frac{\delta f^a[A_\varepsilon]}{\delta A^c_\varepsilon}\frac{\delta A^c_\varepsilon}{\delta\varepsilon^b} = f'^{\,ac}[A_\varepsilon]\frac{1}{g}D^{cb}_\mu$$

Then

$$= \int DA_\varepsilon \, D\varepsilon \mathrm{e}^{\mathrm{i}S[A_\varepsilon]} \det\left(f'^{\,ac}[A_\varepsilon]\right) \det\left(\frac{1}{g}D^{cb}_\mu\right) \delta(f[A_\varepsilon] - F) O_i O_j \cdots$$

hence renaming $A_\varepsilon \to A$ gives

$$= \int D\varepsilon \, DA \, \mathrm{e}^{\mathrm{i}S[A]} \det(f'^{\,ac}[A]) \det\left(\frac{1}{g}D^{cb}_\mu\right) \delta(f[A] - F) O_i O_j \cdots$$

$$= \left(\int D\varepsilon\right) \int DA \, \mathrm{e}^{\mathrm{i}S[A]} \det\left(\frac{\delta f[A_\varepsilon]}{\delta\varepsilon}\right)_{\varepsilon=0} \delta(f[A] - F) O_i O_j \cdots$$

References

[1] Howard M Georgi. *Lie Algebras in Particle Physics*. Frontiers in physics. Cambridge: Perseus, 1999.